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Automated Market Makers and their Implications for Liquidity Providers

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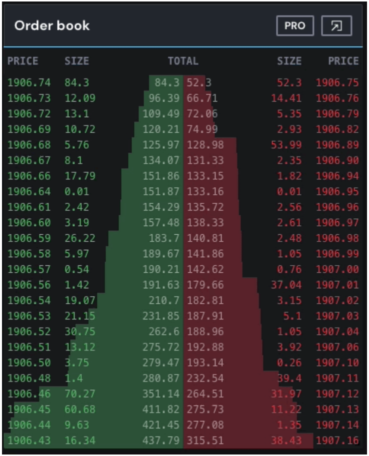
Central Limit Order Book



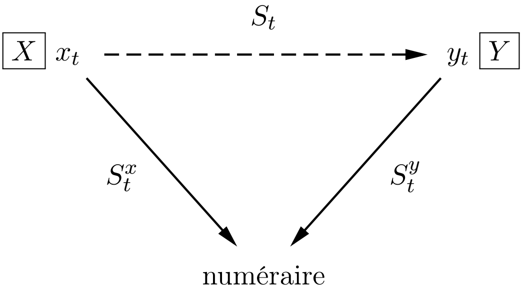
Order book PRO

PRICE	SIZE	TOTAL		SIZE	PRICE
1906.74	84.3	84.3	52.3	52.3	1906.75
1906.73	12.09	96.39	66.71	14.41	1906.76
1906.72	13.1	109.49	72.06	5.35	1906.79
1906.69	10.72	120.21	74.99	2.93	1906.82
1906.68	5.76	125.97	128.98	53.99	1906.89
1906.67	8.1	134.07	131.33	2.35	1906.90
1906.66	17.79	151.86	133.15	1.82	1906.94
1906.64	0.01	151.87	133.16	0.01	1906.95
1906.61	2.42	154.29	135.72	2.56	1906.96
1906.60	3.19	157.48	138.33	2.61	1906.97
1906.59	26.22	183.7	140.81	2.48	1906.98
1906.58	5.97	189.67	141.86	1.05	1906.99
1906.57	0.54	190.21	142.62	0.76	1907.00
1906.56	1.42	191.63	179.66	37.04	1907.01
1906.54	19.07	210.7	182.81	3.15	1907.02
1906.53	21.15	231.85	187.91	5.1	1907.03
1906.52	30.75	262.6	188.96	1.05	1907.04
1906.51	13.12	275.72	192.88	3.92	1907.06
1906.50	3.75	279.47	193.14	0.26	1907.10
1906.48	1.4	280.87	232.54	39.4	1907.11
1906.46	70.27	351.14	264.51	31.97	1907.12
1906.45	60.68	411.82	275.73	11.22	1907.13
1906.44	9.63	421.45	277.08	1.35	1907.14
1906.43	16.34	437.79	315.51	38.43	1907.16

«gif» from https://docs.tuleep.trade/terminal/order-book

The Marketplace

Liquidity Pool at Time t

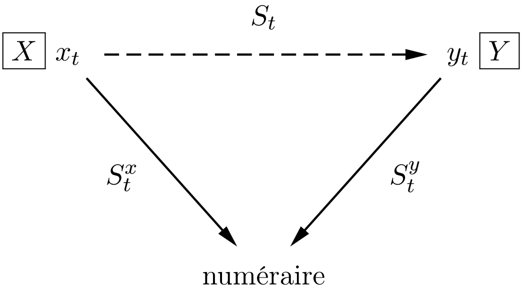


x_t	posted amount of the «liquid» token X (e.g., XTZ, aka «tez»)
y_t	posted amount of the «illiquid» token Y (e.g., YOU)

All quantities with a subscript t represent càdlàg stochastic processes in continuous time.

The Marketplace

Liquidity Pool at Time t



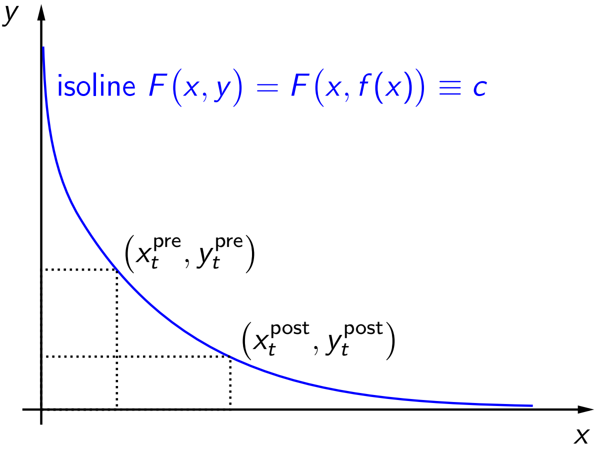
S_t^x exogenously given unit value of X w.r.t. the numéraire (e.g., GBP)
S_t implicit spot FX-rate (e.g., XTZYOU)

Restriction

We assume that $(S_t^x)_{t \geq 0}$ is independent of $(y_t)_{t \geq 0}$.

Automated Market Making (AMM)

Constant Function Market Making (CFMM)



The spot FX-rate satisfies $S_t = \frac{F_x(x_t, y_t)}{F_y(x_t, y_t)} = -f'(x_t)$.

Automated Market Making (AMM)

Features

⊕

self-balancing

low computational demands

«simple»

⊖

capital intensive

returns may be unacceptably bad

«impermanent loss»

Automated Market Making (AMM)

Selected References

- **Systemisation of Knowledge**
  - Mohan, V. (2022)
  - Xu, J., Paruch, K., Cousaert, S., Feng, Y. (2023)
- **Agent-Based Simulation**
  - Sabaté-Vidales, M., Šiška, D. (2022)
  - Cohen, S., Sabaté-Vidales, M., Šiška, D., Szpruch, L. (2023)
- **Impermanent Loss**
  - Loesch, S., Hindman, N., Richardson, M. B. and Welch, B. (2021)
  - Fukasawa, M., Maire, B., Wunsch, M. (2023)
- **AMM beyond CFMM**: Cartea, Á, Drissi, F., Sànchez-Betancourt, L., Šiška, D., Szpruch, L. (2023)
- **Optimal Execution**: Cartea, Á., Drissi, F., Monga, M. (2022)
- **MEV Attacks**: Daian, P., Goldfeder, S., Kell, T., Li, Y., Zhao, X., Bentov, I., Breidenbach, L., Juels, A. (2020)

Automated Market Making (AMM)

Transaction Cost

$$0 \leq \kappa_1 < 1 \quad \text{transaction fee of the infrastructure provider}$$

$$0 \leq \kappa_2 < 1 \quad \text{transaction fee of the liquidity pool}$$

$$0 \leq \kappa := \kappa_1 + \kappa_2 < 1 \quad \text{total transaction fees}$$

E.g., $\kappa_1 = 0.10\%$ and $\kappa_2 = 0.25\%$. However, gross transaction fees earned w.r.t the numéraire are typically lower than 0.35%, ceteris paribus.

Automated Market Making (AMM)

Dynamics of a CFMM when posting Δx

Let us assume that a market participant initiates a swap at time t by posting Δx of X to the liquidity pool. In return, she receives $-\Delta y(\Delta x)$ of Y (the sign is by convention), where $\Delta y(\Delta x) < 0$ solves the equation

$$F\left(x_t^{\text{pre}} + (1-\kappa)\Delta x, y^{\text{pre}} + \Delta y(\Delta x)\right) \stackrel{!}{=} F\left(x_t^{\text{pre}}, y_t^{\text{pre}}\right).$$

One part of the transaction fees is paid implicitly to the pool by increasing the liquidity to the updated isoline

$$\begin{aligned} c_t^{\text{post}}(\Delta x) &:= F\left(x_t^{\text{post}}(\Delta x), y_t^{\text{post}}(\Delta x)\right) \\ &:= F\left(x_t^{\text{pre}} + (1-\kappa_1)\Delta x, y_t^{\text{pre}} + \Delta y(\Delta x)\right). \end{aligned}$$

The residual part $\kappa_1 \Delta x$ is paid out to the infrastructure provider and leaves our consideration.

Automated Market Making (AMM)

Dynamics of a CFMM when posting Δy

Analogously, if Δy of Y is posted to the liquidity pool for $-\Delta x(\Delta y)$ of X in return, the corresponding equation reads

$$F\big(x_t^{\text{pre}} + \Delta x(\Delta y), y^{\text{pre}} + (1-\kappa)\Delta y\big) \overset{!}{=} F\big(x_t^{\text{pre}}, y_t^{\text{pre}}\big).$$

Similarly,

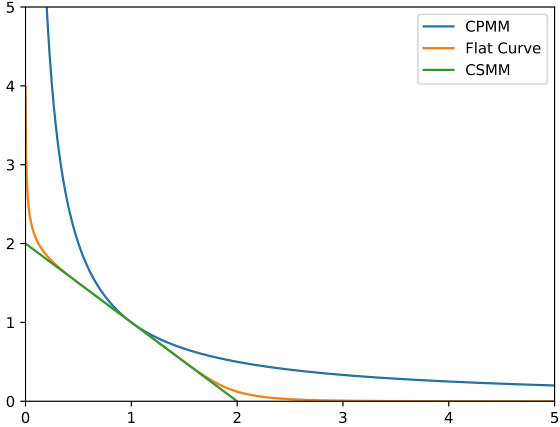
$$\begin{aligned} c_t^{\text{post}}(\Delta y) &:= F\big(x_t^{\text{post}}(\Delta y), y_t^{\text{post}}(\Delta y)\big) \\ &:= F\big(x_t^{\text{pre}} + \Delta x(\Delta y), y_t^{\text{pre}} + (1-\kappa_1)\Delta y\big) \end{aligned}$$

refers to as the updated liquidity level.

A popular choice is the **CPMM** $F(x, y) := xy$ with $S_t = \frac{y_t}{x_t}$.

Digression: Flat Curve

$$F(x, y) = (x + y)^8 - (x - y)^8, \qquad S_t = \frac{(x_t + y_t)^7 - (x_t - y_t)^7}{(x_t + y_t)^7 + (x_t - y_t)^7}$$



CPMM: Market Capitalisation versus Sale Value

Let us assume that one holds a fraction $\lambda \in [0, 1]$ of the liquidity pool. The **accounting value** of the holding amounts to

$$V_t = 2\lambda S_t^x x_t. \tag{1}$$

In turn, the **sale value** can be calculated by withdrawing λx_t and λy_t – leading to a new liquidity of $(1-\lambda)^2 x_t y_t$ – and swapping $\Delta y = \lambda y_t$ against*

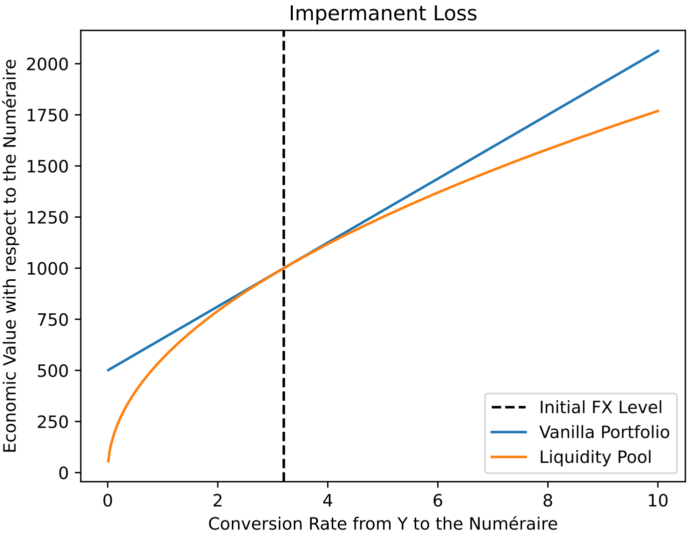
$$\Delta x(\Delta y) = \frac{(1-\kappa)(1-\lambda)\lambda x_t}{1-\kappa\lambda}. \tag{2}$$

With respect to the numéraire, this leaves us with

$$\widetilde{V}_t = \frac{\lambda(2-\kappa-\lambda)}{1-\kappa\lambda} S_t^x x_t = \underbrace{2\lambda S_t^x x_t}_{=V_t} - \underbrace{\frac{\lambda(\kappa+\lambda-2\kappa\lambda)}{1-\kappa\lambda} S_t^x x_t}_{\text{illiquidity premium}}. \tag{3}$$

* $\left[(1-\lambda)x - \Delta x(\Delta y)\right]\left[(1-\lambda)y + (1-\kappa)\lambda y\right] \overset{!}{=} (1-\lambda)^2 xy$

CPMM: Impermanent Loss



CPMM: Valuation of the Impermanent Loss

For simplicity, let $\kappa = 0$. Let us assume that we provided x_t and y_t to the liquidity pool at time t, which was worth

$$S_t^x x_t + S_t^y y_t = S_t^x (x_t + {S_t}^{-1} y_t) = 2 S_t^x x_t$$

with respect to the numéraire. If the FX-rate at time $t + \Delta t$ changed to $S_{t+\Delta t}$, then arbitrageurs would have rebalanced the amounts to

$$x_{t+\Delta t} = \sqrt{\frac{c_t}{S_{t+\Delta t}}}, \qquad y_{t+\Delta t} = \sqrt{c_t S_{t+\Delta t}}.$$

Therefore, the opportunity cost would amount to

$$\underbrace{S_{t+\Delta t}^x \left(x_{t+\Delta t} + {S_{t+\Delta t}}^{-1} y_{t+\Delta t}\right)}_{\text{rebalanced liquidity pool}} - \underbrace{S_{t+\Delta t}^x \left(x_t + {S_{t+\Delta t}}^{-1} y_t\right)}_{\text{vanilla portfolio}}$$

$$= -S_{t+\Delta t}^x x_t \left(1 - \sqrt{\frac{S_t}{S_{t+\Delta t}}}\right)^2.$$

CPMM: Valuation of the Impermanent Loss

By the general *Carr-Madan-formula*, it holds for any $g : (0, \infty) \longrightarrow \mathbb{R}$ subject to sufficient regularity

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \int_0^{x_0} g''(k) \max\{k - x, 0\}\, dk + \int_{x_0}^{\infty} g''(k) \max\{x - k, 0\}\, dk.$$

For the impermanent loss measured in X, it holds at $S_t = \frac{y_t}{x_t}$

$$\ell^X(s) = \left(\sqrt{x_t} - \sqrt{\frac{y_t}{s}}\right)^2 \qquad \ell^X(S_t) = 0,$$

$$\partial_s \ell^X(s) = \sqrt{\frac{x_t y_t}{s^3}} - \frac{y_t}{s^2} \qquad \partial_s \ell^X(S_t) = 0,$$

$$\partial_{ss} \ell^X(s) = -\frac{3}{2}\sqrt{\frac{x_t y_t}{s^5}} + \frac{2y_t}{s^3}.$$

CPMM: Model-Independent Simulation

Let $p \in (0, \infty)$ and $S_t^{\text{post}} = pS_t^{\text{pre}}$. Let us introduce the auxiliary quantity

$$\xi(\kappa_1, \kappa_2, p) := \frac{-p(2 - 2\kappa_1 - \kappa_2) + \sqrt{p^2\kappa_2{}^2 + 4p(1-\kappa_1)(1-\kappa_1-\kappa_2)}}{2p(1-\kappa_1)(1-\kappa_1-\kappa_2)}.$$

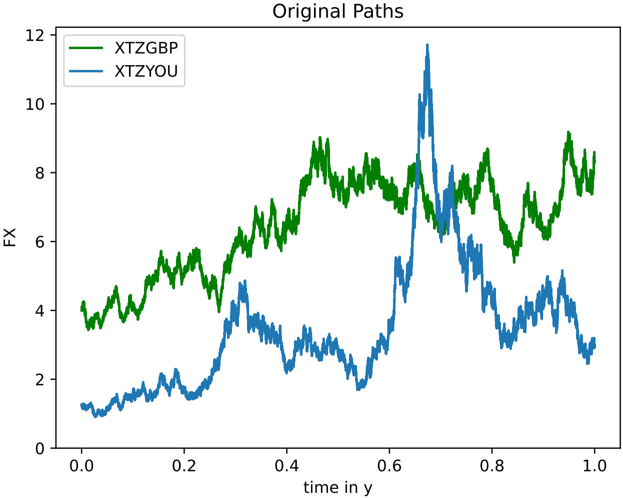
Note that $\xi(0, 0, p) = \frac{\sqrt{p}}{p} - 1$ and $\xi(\kappa_1, \kappa_2, 1) = 0$. Then, one can retrieve the posted amounts

$$\begin{cases} \Delta x(p) = \xi(\kappa_1, \kappa_2, p) x_t^{\text{pre}} & \text{if } p \leq 1, \\ \Delta y(p) = \xi(\kappa_1, \kappa_2, p^{-1}) y_t^{\text{pre}} & \text{if } p > 1, \end{cases}$$

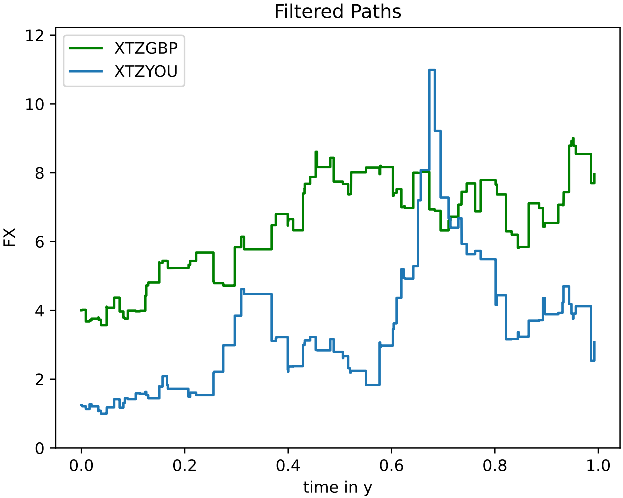
as well as the revaluation of the liquidity pool

$$V_t^{\text{post}}(p) = \begin{cases} 2S_t^x x_t^{\text{pre}} \big(1 + (1-\kappa_1)\xi(\kappa_1, \kappa_2, p)\big) & \text{if } p \leq 1, \\ \dfrac{2S_t^x x_t^{\text{pre}}}{\big(1 + (1-\kappa)\xi(\kappa_1, \kappa_2, p^{-1})\big)} & \text{if } p > 1. \end{cases}$$

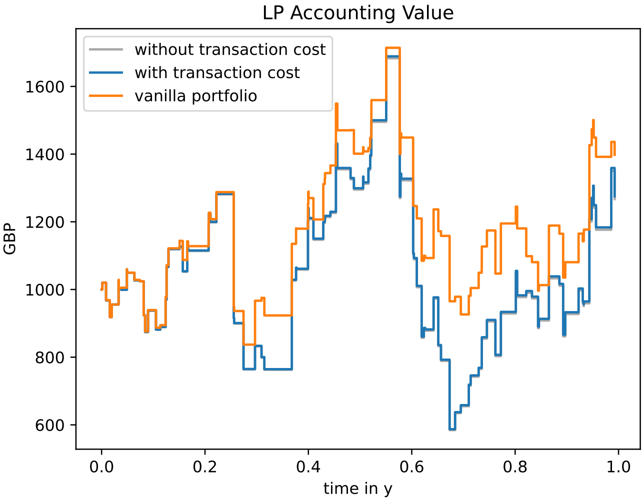
CPMM: Model-Independent Simulation



CPMM: Model-Independent Simulation



CPMM: Model-Independent Simulation



CPMM: Break-Even Transaction Cost

Proposition (Break-Even Transaction Cost)

Let us consider a CPMM. If one establishes order size and pool size dependent transaction fees

$$\kappa_2(\Delta x) = \frac{(1-\kappa_1)^2}{\frac{x_t^{\text{pre}}}{\Delta x} + (1-\kappa_1)}, \qquad \kappa_2(\Delta y) = \frac{(1-\kappa_1)^2}{\frac{y_t^{\text{pre}}}{\Delta y} + (1-\kappa_1)},$$

then the liquidity providers do not incur an impermanent loss. It obviously holds $\lim_{\Delta x \to \infty} \kappa_2(\Delta x) = \lim_{\Delta y \to \infty} \kappa_2(\Delta y) = 1 - \kappa_1$.

CPMM: Break-Even Transaction Cost

Proof

Posting $\Delta x > 0$ yields to

$$V_t^{\text{post}}(\Delta x) \overset{!}{=} S_t^x\big(x_t^{\text{pre}} + S_t^{\text{post}}(\Delta x)^{-1} y_t^{\text{pre}}\big),$$

where

$$V_t^{\text{post}}(\Delta x) = 2S_t^x\big(x_t^{\text{pre}} + (1-\kappa_1)\Delta x\big),$$

$$S_t^{\text{post}}(\Delta x)^{-1} = \frac{\big(x_t^{\text{pre}} + (1-\kappa_1)\Delta x\big)\Big(x_t^{\text{pre}} + \big(1-\kappa_1-\kappa_2(\Delta x)\big)\Delta x\Big)}{x_t^{\text{pre}} y_t^{\text{pre}}}.$$

$$\Longleftrightarrow \qquad \kappa_2(\Delta x) = \frac{(1-\kappa_1)^2 \Delta x}{x_t^{\text{pre}} + (1-\kappa_1)\Delta x} = \frac{(1-\kappa_1)^2}{\frac{x_t^{\text{pre}}}{\Delta x} + (1-\kappa_1)}.$$

The argument works analogously for posting $\Delta y > 0$. $\square$

CPMM: Optimal Trade Execution

When posting $\Delta x > 0$ twice, the liquidity pool still ends up with

$$\big(x_t^{\text{post}}(\Delta x)\big)(\Delta x) = x_t^{\text{pre}} + 2(1-\kappa_1)\Delta x = x_t^{\text{post}}(2\Delta x),$$

whereas

$$\begin{aligned}
\big(y_t^{\text{post}}(\Delta x)\big)(\Delta x) &= \frac{\big(x_t^{\text{pre}} + (1-\kappa_1)\Delta x\big)\dfrac{x_t^{\text{pre}} y_t^{\text{pre}}}{x_t^{\text{pre}} + (1-\kappa)\Delta x}}{\big(x_t^{\text{pre}} + (1-\kappa_1)\Delta x\big) + (1-\kappa)\Delta x} \\
&= \frac{x_t^{\text{pre}} + (1-\kappa_1)\Delta x}{x_t^{\text{pre}} + (1-\kappa)\Delta x} \cdot \frac{x_t^{\text{pre}} y_t^{\text{pre}}}{\kappa_2 \Delta x + x_t^{\text{pre}} + (1-\kappa)2\Delta x} \\
&> \underbrace{\frac{x_t^{\text{pre}} + (1-\kappa_1)\Delta x}{x_t^{\text{pre}} + (1-\kappa)\Delta x + \kappa_2 \Delta x}}_{=1} \cdot \frac{x_t^{\text{pre}} y_t^{\text{pre}}}{x_t^{\text{pre}} + (1-\kappa)2\Delta x} \\
&= y_t^{\text{post}}(2\Delta x).
\end{aligned}$$

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