Local Volatility Models for Commodity Forwards

Based on ongoing work with Nils Detering

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WPI-Workshop Wien
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THE ENERGY FORWARDS

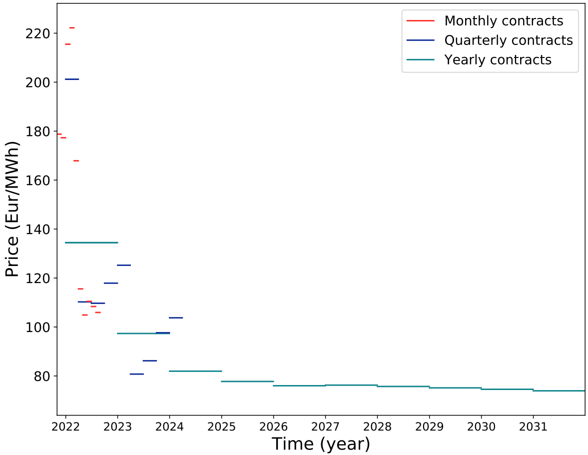
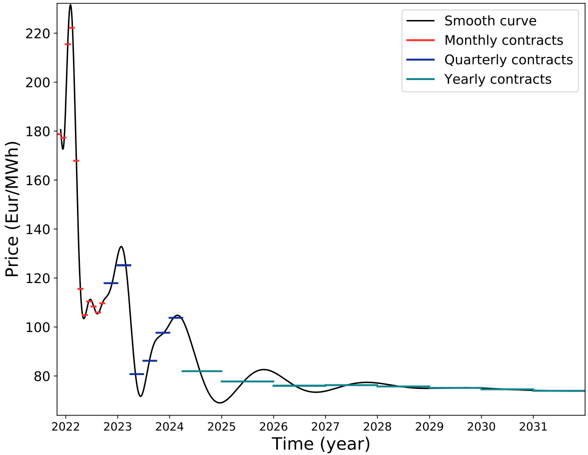


Figure: EEX German Power Future prices on November 26, 2021.

THE ENERGY FORWARDS



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THE VOLATILITY SMILE FOR ENERGY OPTIONS

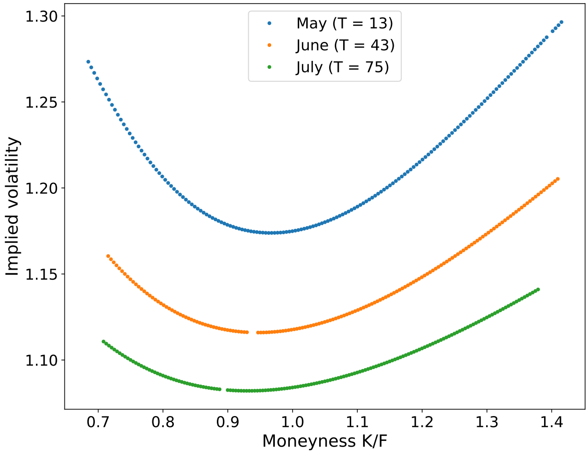

Figure: EEX German Power Options implied volatility on April 13, 2022.

OUTLINE

The forward curve in energy markets

No-arbitrage conditions

Motivating examples

Local volatility models for forward curves

Map ahead: model calibration

OUTLINE

THE ENERGY FORWARDS

- Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$;
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- We denote by $F(t, T_1, T_2)$ the t-time price of a forward with delivery in $[T_1, T_2]$;
- We are interested in (call) options on forward contracts with delivery period:

$$\Pi(t) = e^{-\int_t^T r(s)ds} \mathbb{E}_{\mathbb{Q}} \left[\pi \left(F(T, T_1, T_2) \right) \middle| \mathcal{F}_t \right] \tag{1}$$

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- We work with the Musiela parametrization

$$g(t, x) := F(t, t + x)$$

where $x := T - t$ is the time to maturity.

We need:

- Stochastic volatility;
- A model flexible enough to recover the volatility smile;
- To incorporate the maturity effect (Samuelson effect);
- Seasonality.

The class of models that we are going to present can be used for different commodity markets.

THE FORWARD DYNAMICS

We model $g_t := g(t, \cdot)$ as an element in a suitable Hilbert space $\mathcal{H}$.

We consider the following Stochastic Partial Differential Equation (SPDE):

$$dg_t = \partial_x g_t dt + a(t, g_t)dt + \sigma(t, g_t)d\mathbb{W}_t, \tag{3}$$

where

- ∂_x is the generator for the shift-semigroup $\mathcal{S}_t f(x) = f(t + x)$;
- $\{\mathbb{W}_t\}_{t \geq 0}$ is an $\mathcal{H}$-valued Wiener process with covariance operator $\mathcal{Q} \in \mathcal{L}(\mathcal{H})$;
- $a : \mathbb{R}_+ \times \mathcal{H} \to \mathcal{H}$ and $\sigma : \mathbb{R}_+ \times \mathcal{H} \to \mathcal{L}(\mathcal{H})$ are state-dependent coefficients.

Equation (3) is an instance of a local volatility model for forward curves, where the drift a and the volatility operator σ are functions of time and the current level of the forward curve g.

THE FORWARD CURVE SPACE $\mathcal{H}$

We model g_t as an element in the Filipović space:

For a given continuous and non-decreasing function $\alpha : \mathbb{R}_+ \to [1, \infty)$ with $\alpha(0) = 1$, $\mathcal{H}$ is the Hilbert space of all absolutely continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ for which

$$\int_{\mathbb{R}_+} f'(x)^2 \alpha(x) dx < \infty \tag{4}$$

with the inner product

$$\langle f_1, f_2 \rangle := f_1(0) f_2(0) + \int_{\mathbb{R}_+} f_1'(x) f_2'(x) \alpha(x) dx \quad \text{for } f_1, f_2 \in \mathcal{H},$$

and norm $\|f_1\|^2 := \langle f_1, f_1 \rangle = f_1(0)^2 + \int_{\mathbb{R}_+} f_1'(x)^2 \alpha(x) dx$.

Let: $\mathcal{H}^+ := \{f \in \mathcal{H} : f(x) \geq 0 \text{ for every } x \in \mathbb{R}_+\}$.

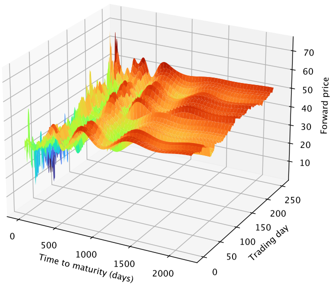

Figure: Smooth forward curves from 02/01/2020 to 30/12/2020.

OUTLINE

NO ARBITRAGE FOR ENERGY FORWARDS

- The forwards $F(t, T_1, T_2)$ are traded asset, hence their (discounted) value must a martingale under an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$;

- In particular, if $F(t, T)$ is a martingale under an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, then the previous point is satisfied because of equation (2);

- However, this is a sufficient, but, strictly speaking, not necessary condition: one may be interested in only *some* forward contracts.

- Also: it shouldn't be possible to construct arbitrage opportunity by trading on overlapping contracts. *This already holds by construction* (2).

CHANGE OF MEASURE: PRELIMINARIES

- We know that, if a stochastic process $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ satisfies *some conditions*, then we can define a measure $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{F})$ via

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_{\bar{T}}} = \exp\left\{\int_0^{\bar{T}} \langle \varphi(s), d\mathbb{W}_s \rangle - \frac{1}{2}\int_0^{\bar{T}} |\varphi(s)|^2 \, ds\right\}. \tag{5}$$

- The process $\tilde{\mathbb{W}}$ defined by

$$\tilde{\mathbb{W}}_t := \mathbb{W}_t - \int_0^t \varphi(s) ds \tag{6}$$

is a $\mathcal{Q}$-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{Q})$.

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Theorem (Part 1)

Let the process $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *be defined by*

$$\sigma(t, g_t)\varphi(t) = r(t)g_t - a(t, g_t) \quad \text{for all } t \geq 0, \tag{7}$$

and satisfy "those conditions". Then for $\{\tilde{W}_t\}_{t\geq 0}$ *defined in* (6)*,* g_t *has the following dynamics under* $\mathbb{Q}$*:*

$$dg_t = \partial_x g_t dt + r(t)g_t dt + \sigma(t, g_t)d\tilde{\mathbb{W}}_t. \tag{8}$$

In particular, if σ *is of linear growth, then the process*

$$e^{-\int_0^t r(s)ds}F(t, T) = e^{-\int_0^t r(s)ds}g_t(T - t)$$

is a real-valued martingale under $\mathbb{Q}$.

CHANGE OF MEASURE (2)

Theorem (Part 2)

Let $\mathfrak{B} := \{B = [b_1, b_2] : b_2 > b_1 > 0]\}$. *Let the process* $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$\int_B \sigma(t, g_t)\varphi(t)(T - t)dT = \int_B (r(t)g_t - a(s, g_t))(T - t)dT \qquad \text{for all } B \in \mathfrak{B}, \tag{9}$$

and such that "those conditions" are satisfied. Then for $\{\tilde{W}_t\}_{t \geq 0}$ *defined in* (6), g_t *has the following dynamics under* $\mathbb{Q}$:

$$dg_t = \partial_x g_t dt + (a(t, g_t) + \sigma(t, g_t)\varphi(t))\, dt + \sigma(t, g_t) d\tilde{\mathbb{W}}_t. \tag{10}$$

In particular, if σ *is of linear growth, then for every* $[b_1, b_2] = B \in \mathfrak{B}$, *the process*

$$e^{-\int_0^t r(s)ds} F(t, b_1, b_2) = \frac{e^{-\int_0^t r(s)ds}}{b_2 - b_1} \int_{b_1}^{b_2} g_t(T - t)dT$$

is a real-valued martingale under $\mathbb{Q}$.

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THE CEV SPECIFICATION

The first specification we had in mind is the CEV model:

$$dg_t = \partial_x g_t dt + \beta_t g_t^{\gamma} d\mathbb{W}_t, \qquad \gamma \geq 1, \tag{11}$$

with $\beta : \mathbb{R}_+ \to \mathbb{R}$ modeling e.g. the seasonality/time-dependence.

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with $\beta : \mathbb{R}_+ \to \mathbb{R}$ modeling e.g. the seasonality/time-dependence. Why not $\gamma < 1$? What makes this problematic is the $\mathcal{H}$-norm:

$$\|f^{\gamma}\|^2 = (f^{\gamma}(0))^2 + \int_0^{\infty} \left(\frac{\partial f(x)}{\partial x} \gamma f^{\gamma-1}(x) \right)^2 \alpha(x) dx.$$

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We need to make sure that the map

$$\Psi : \mathcal{H} \to \mathcal{H}$$
$$f \mapsto \Psi(f) := f^{\gamma}$$

1. ...is well defined in $\mathcal{H}$;
2. ...is **locally bounded** and **locally Lipschitz** w.r.t. the $\mathcal{H}$-norm;

THE CEV IMPLIED VOLATILITY

3. It turns out that the CEV model is not very flexible:

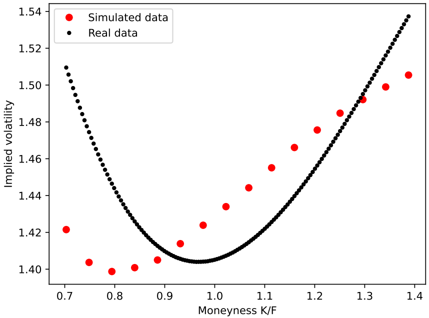

Figure: Example of implied volatility in the CEV model for $\gamma = 1.4$[1].

[1] The data was simulated with a flat initial curve and one noise factor.

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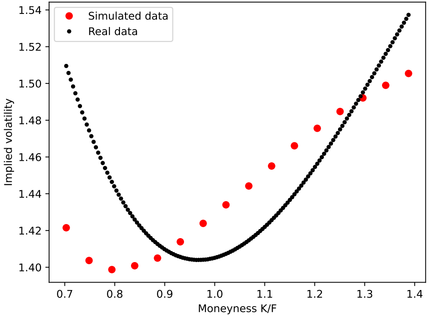

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4. The CEV is also numerically unstable (intrinsic explosive nature).

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Motivated, e.g. by [Ingersoll (1996)] and [Schlenkrich (2018)], we consider a *spline-type of specification:*

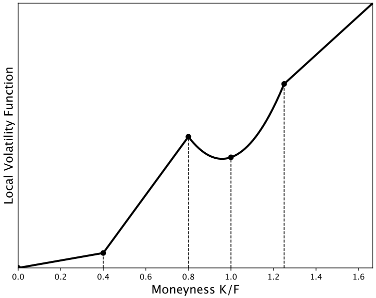

Figure: Example of local volatility function.

where the spline is quadratic around the ATM values and linear otherwise.

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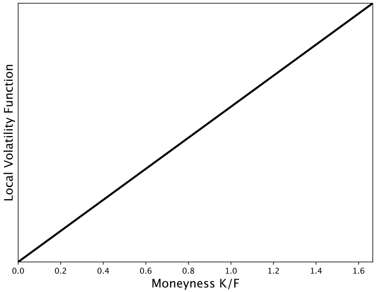

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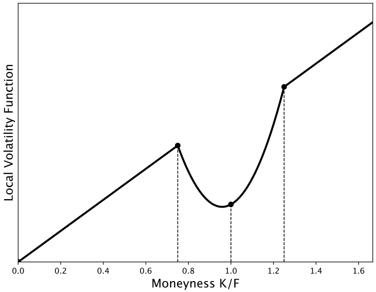

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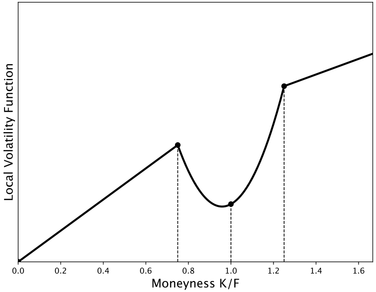

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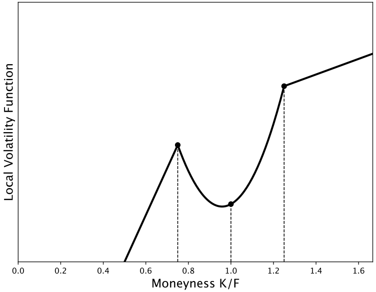

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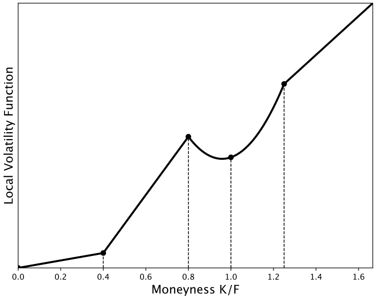

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We need an extra ”internal” point to avoid negative volatility and sufficiently fast decay.

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NEW LOCAL VOLATILITY MODEL AT WORK

The new model is able to capture the smile observed in the energy markets[2]:

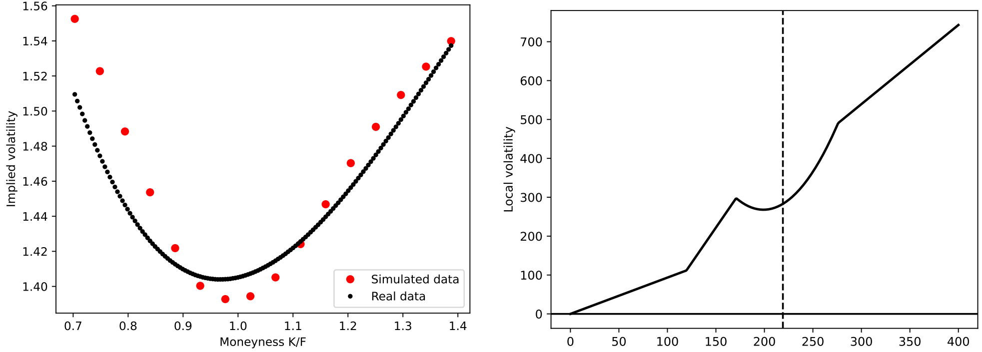

Figure: Left: Implied volatility on real data vs implied volatility on simulated data with the spline-type local volatility specification (where the parameters have been *manually* calibrated). Right: the local volatility function.

[2]The data was simulated with a flat initial curve and one noise factor.

THE MODEL

The new specification is now:

$$dg_t = \partial_x g_t dt + \beta_t \, \sigma_{\mathrm{sp}}(g_t) d\mathbb{W}_t, \tag{12}$$

where $\sigma_{\mathrm{sp}}(f)$ is of *spline-type* in the sense discussed before.

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Again, given $s : \mathbb{R} \to \mathbb{R}$ a spline function, we need to make sure that the map

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$$f \mapsto \Psi(f) := s \circ f$$

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However:

3. We need to work with weakly differentiable maps (unless to smooth the "edges" – also possible).

OUTLINE

LOCAL VOLATILITY MODELS

The diffusion term is of the form

$$\begin{aligned}\sigma : \mathbb{R}_+ \times \mathcal{H} &\to \mathcal{L}(\mathcal{H})\\ (t,f) &\mapsto \sigma(t,f).\end{aligned}$$

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- We need a rich class of operators in $\mathcal{L}(\mathcal{H})$:
  - Every function $\phi \in \mathcal{H}$ defines an element $\mathcal{M}_\phi \in \mathcal{L}(\mathcal{H})$ by

$$\mathcal{M}_\phi(h) := \phi h;$$

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- We need a sufficiently rich class of functions from $(\mathbb{R}_+ \times)\mathcal{H}$ to $\mathcal{H}$:
  - Suitable maps are those that act point-wise on the curve

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$$\Psi(t, f)(x) = \psi(t, f(x)) \, .$$

- We will then consider σ's of the form $\sigma(t, f) = \mathcal{M}_{\psi_t \circ f} = \mathcal{M}_{\Psi(t,f)}$.

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Let us for now drop the time-dependency.

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Remember the $\mathcal{H}$ norm:

$$\|\psi \circ f\|^2 = ((\psi \circ f)(0))^2 + \int_0^\infty \left(\frac{\partial f}{\partial x}(x)(\psi' \circ f)(x)\right)^2 \alpha(x)dx.$$

This norm interferes with the locally bounded conditions, locally Lipschitz conditions, etc...

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This norm interferes with the locally bounded conditions, locally Lipschitz conditions, etc...

We then must analyze under which conditions a function $\psi : \mathbb{R} \to \mathbb{R}$ defines a map

$$\begin{aligned} \Psi :\ & \mathcal{H} \to \mathcal{H} \\ & f \mapsto \Psi(f) := \psi \circ f = \{x \mapsto \psi(f(x))\}. \end{aligned}$$

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A FIRST RESULT

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We want to analyze under which conditions $\psi : \mathbb{R} \to \mathbb{R}$ defines a map

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Proposition

- *If $\psi : \mathbb{R} \to \mathbb{R}$ has continuous derivative and $f \in \mathcal{H}$, then $\psi \circ f \in \mathcal{H}$;*
- *If $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ has continuous derivative and $\lim_{x \to 0} |\psi'(x)| < \infty$, for $f \in \mathcal{H}^+$, it holds $\psi \circ f \in \mathcal{H}^+$.*

In both cases, there exist constants $M_n > 0$ such that $\|\psi \circ f\| \leq M_n$ for every $f \in \mathcal{H}$ with $\|f\| \leq n$, i.e. Ψ is locally bounded.

A SECOND RESULT

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Proposition

If $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, (t, y) \mapsto \psi(t, y)$ *has locally Lipschitz derivative in the second argument, then the map*

$$\begin{aligned} \Psi : \mathbb{R}_+ \times \mathcal{H} &\to \mathcal{H} \\ (t, f) &\mapsto \Psi(t, f) := \{x \mapsto \psi(t, f(x))\} \end{aligned}$$

is locally Lipschitz for any $f \in \mathcal{H}$*.*
The same also holds for $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *and* $f \in \mathcal{H}_+$ *.*

In particular, the map

$$\begin{aligned} \mathbb{R}_+ \times \mathcal{H} &\to \mathcal{L}(\mathcal{H}) \\ (t, f) &\mapsto \mathcal{M}_{\Psi(t,f)} \end{aligned}$$

is locally Lipschitz.

OUR SPECIFICATIONS

- CEV specification with $\gamma \in \mathbb{R}^+$:
  - Well defined for every $\gamma \geq 1$;
  - Locally bounded for every $\gamma \geq 1$;
  - Locally Lipschitz for every $\gamma \geq 2$;

- Spline specification:
  - Bounded;
  - Well defined and locally Lipschitz if the spline is C^1;

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  - Well defined for every $\gamma \geq 1$;
  - Locally bounded for every $\gamma \geq 1$;
  - Locally Lipschitz for every $\gamma \geq 2$;
  - For $1 < \gamma < 2$ we need to consider a "functional" γ so to deal with the issues for $x \to 0$.
- Spline specification:
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  - Locally bounded for every $\gamma \geq 1$;
  - Locally Lipschitz for every $\gamma \geq 2$;
  - For $1 < \gamma < 2$ we need to consider a "functional" γ so to deal with the issues for $x \to 0$.
- Spline specification:
  - Bounded;
  - Well defined and locally Lipschitz if the spline is C^1;
  - For C^0-splines we need to deal with the knots (where the classical derivative is not defined): still possible.

GEOMETRIC MODELS

We now want to consider an exponential class of models.

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For this we need first to make sure that:

- The map $f \mapsto \{x \mapsto e^{f(x)}\}$ is well defined from $\mathcal{H}$ to $\mathcal{H}$;
- The map $h \mapsto \{x \mapsto \log(h(x))\}$ is well defined from $\mathcal{H}$ to $\mathcal{H}$.

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We can prove that:

- The map $f \mapsto \{x \mapsto e^{f(x)}\}$ is well defined from $\mathcal{H}$ to $\mathcal{H}_>$ (this was shown in [Benth and Krühner (2015)] but also corollary to our first proposition);
- The map $h \mapsto \{x \mapsto \log(h(x))\}$ is well defined from $\mathcal{H}_>$ to $\mathcal{H}$;

where

$$\mathcal{H}_> := \{h_+ \in \mathcal{H} | h(x) > 0 \; \forall x, \lim_{x \to \infty} h(x) > 0\}.$$

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In particular:

- $\log(e^h) = h$ for $h \in \mathcal{H}$, i.e. log acts as a left inverse to e on $\mathcal{H}$;
- $e^{\log h} = h$ for $h \in \mathcal{H}_>$, i.e. e acts as a left inverse to log on $\mathcal{H}_>$;

and one can recover by Itô's formula the dynamics of e^{g_t} starting from the dynamics of g_t and vice-versa.

OUTLINE

The forward curve in energy markets

No-arbitrage conditions

Motivating examples

Local volatility models for forward curves

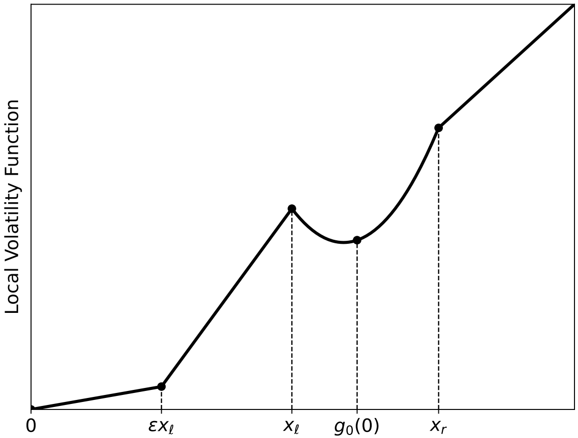
Map ahead: model calibration

PARAMETRIZATION OF THE LOCAL VOLATILITY FUNCTION

- $0.5 < \omega_\ell \leq 1$
- $1 < \omega_r \leq 1.5$
- $\beta_\ell \in I_\ell \subset \mathbb{R}_+$
- $\beta_r \in I_r \subset \mathbb{R}_+$
- $0 < \epsilon < 1$
- $\sigma_{\text{ATM}}(x) = ax^2 + bx + c$

Then:

$$\sigma_{\text{sp}}(x) = \begin{cases} \beta_\ell^0 x & 0 \leq x < \epsilon x_\ell \\ \alpha_\ell + \beta_\ell x & \epsilon x_\ell \leq x < x_\ell \\ \sigma_{\text{ATM}}(x) & x_\ell \leq x < x_r \\ \alpha_r + \beta_r x & x \geq x_r \end{cases}$$



where $x_\ell := \omega_\ell g_0(0)$ and $x_r := \omega_r g_0(0)$.

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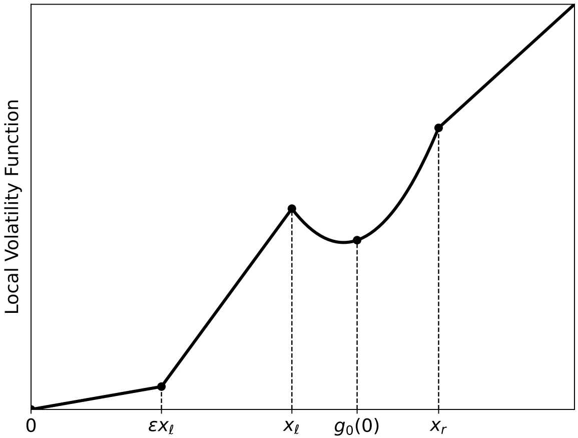

where $x_\ell := \omega_\ell g_0(0)$ and $x_r := \omega_r g_0(0)$.

We then also have parameters for the covariance structure, etc..

THE (HIGHLY COSTLY) SIMULATIONS

For every $g_t \in \mathcal{H}$ and $T \geq t$, we have

$$g_T = \mathcal{S}_{T-t} g_t + \int_t^T \mathcal{S}_{T-u} \sigma(u, g_u) d\mathbb{W}_u. \tag{13}$$

Let now

- $\{e_j\}_j$ orthonormal functions in $\mathcal{H}$;
- $\{c_j\}_j$ strictly positive numbers;
- β_j mutually independent BMs.

Then:

$$g_T = \mathcal{S}_{T-t} g_t + \sum_j \sqrt{c_j} \int_t^T \mathcal{S}_{T-u} \sigma(u, g_u) e_j d\beta_u^j. \tag{14}$$

Example with one noise factor:

$$g_T(x) = g_t(T - t + x) + \sqrt{c_1} \int_t^T \sigma\left(u, g_u(T - t + x)\right) e_1(T - t + x) d\beta_u^1.$$

THE TWO-STEP APPROACH [Benth, Detering, L. (2021)]

BI

Let $\lambda \in \Lambda \subset \mathbb{R}^M$ (e.g. $\lambda = (K, T, T_1, T_2)$) be some contract specifications, and $\{\Pi_i\}_{i=1}^N$ some market observed prices $\{\Pi_i\}_{i=1}^N$ corresponding to $\{\lambda_i\}_{i=1}^N$. Then:

1. We train a neural network

$$\mathcal{N}(\cdot, \mathcal{P}) : I \to \mathbb{R}_+^N$$

   to approximate the pricing functional $\Pi_t = \Pi_t(\lambda, \theta)$:

$$\hat{\mathcal{P}} \in \underset{\mathcal{P}}{\operatorname{argmin}} \frac{1}{N_{train}} \frac{1}{N} \sum_{j=1}^{N_{train}} \sum_{i=1}^{N} \left(\mathcal{N}(\theta_j, \mathcal{P})_i - \Pi_t(\theta_j, \lambda_i)\right)^2 .$$

2. We use $\hat{\mathcal{N}}(\theta) := \mathcal{N}(\theta, \hat{\mathcal{P}})$ in calibration to fit market data $\{\Pi_i\}_{i=1}^N$:

$$\hat{\theta} \in \underset{\theta \in I}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\mathcal{N}}(\theta)_i - \Pi_i\right)^2 .$$

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We still need to simulate trajectories of g_t to create the training set in step 1: we use the training routine proposed by [Benth et al. (2022)].

CONCLUSIONS

- We study local-volatility models in the Filipović space for instantaneous forward curves;
- This requires particular care because of the nature of the Filipović norm;
- These serve as models for pricing options in the energy markets;
- With the [Benth et al. (2022)] routine, we train a neural network to approximate the pricing functional;
- Adapting [Benth, Detering, L. (2021)] we use the network for model calibration;
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Thanks for the attention!

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