A McKean-Vlasov game of commodity production, consumption and trading

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joint work with

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Outline

The interaction model

We consider a market where a **producer** interacts with a **processor (consumer)** who buys some commodity and transforms it into a final good (e.g. crude oil into gasoline, wheat into bread)

	Controls (drift and vola)	**Impact on spot price S**	**Forward contract λ units, price F**
Producer	production rate q	negative	short
Consumer	consumption rate c	positive	long

New:

- Risk aversion towards financial position: via an integrated-variance penalization $\Rightarrow$ linear-quadratic McKean-Vlasov (MKV) game
- (Agreement) indifference price of the commodity

Aim: complete description of Nash equilibrium and study of the effect on the forward price of risk aversions and vola controlling costs.

Mean-field and McKean-Vlasov literature

- **Mean-field modeling of interacting economic agents:** [Lasry and Lions (2006), Lasry and Lions (2006a), Huang et al. (2006)].
- **MKV Games:** zero-sum case [Cosso and Pham (2019)]; linear-quadratic case [Miller and Pham (2018)] and [Basei and Pham (2019)]; with terminal constraint [Fu and Horst(2020)].
- **MKV model for energy markets:** [Aïd et. al (2020)].

Application to Economics of games with finitely many actors and MKV dynamics and obj functionals is very recent!

The agenda - Mathematics findings

1. Find a **Nash equilibrium** (for fixed volume λ and price F of the fwd contract), with semi-explicit expressions for equilibrium strategies and payoffs.
2. Compute the indifference price for each player (as a function of λ), induced by the (Nash) equilibrium strategies.
3. Look for the trading volume λ such that the players agree on the forward price $\rightsquigarrow$ **Agreement indifference price**.
4. Study how parameters affect the agreement indifference price and the trading volume.

The agenda - Economics findings

1. The forward agreement indifference price is higher (resp. lower) than the expected spot price when the producer is more (resp. less) risk-averse than the consumer. Speculators (to enter in the agreement): a seller requires a higher forward price and a buyer asks for a lower price.
2. The presence of market power of both players allows for the formation of an equilibrium. Consistency with hedging pressure theory applied to a market populated with producers and consumers acting as speculators.
3. Producers can achieve the same agreement indifference price and the same trading volume either by having high risk aversion and a low volatility control cost, or a low risk aversion and a high volatility control cost.

Literature on indifference pricing and risk premium

- **Indifference pricing:**
  [Henderson and Hobson(2009)] for a survey, [Benth et. al (2008)] for energy markets.

- **Formation of the Risk Premium[1] (commodity):**
  - Normal backwardation theory [Keynes (1930)]: fwd price lower than expected spot price.
  - Hedging pressure theory: the risk premium is determined by the relative risk aversion of producers and consumers (traders) [De Roon et. al (2000), Hirschleifer (1988), Hirschleifer (1988a), Hirschleifer (1990), Ekeland et. al (2019)].

[1] Difference unitary agreement price-expected spot price.

The state variables and players' strategies

- Production rate: $\{u_t\}_{t\in[0,T]}$ and $\{z_t\}_{t\in[0,T]}$ are the strategies

$$dq_t = u_t dt + z_t dW_t, \quad q_0 > 0;$$

- Consumption rate: $\{v_t\}_{t\in[0,T]}$ and $\{y_t\}_{t\in[0,T]}$ are the strategies

$$dc_t = v_t dt + y_t dB_t, \quad c_0 > 0.$$

- Observed market price (linear impact):

$$S_t := s_0 - \rho_p q_t + \rho_c \gamma c_t, \qquad s_0 > 0 \text{ and } \rho_p, \rho_c > 0.$$

- Admissible strategies: $\mathcal{A}^2 := \mathcal{A} \times \mathcal{A}$, where $\mathcal{A} = L^2_{\mathbb{F}}(\Omega \times [0,T], \mathbb{R}^2)$.

N.B. W and B independent. Interaction? Via the financial derivative!

Cumulative profits

$$P_T^p := \int_0^T \overbrace{q_t S_t}^{\text{profit from production}} \underbrace{- \frac{k_p}{2} u_t^2 - \frac{\ell_p}{2}(z_t - \sigma_p)^2}_{\text{costs of controls}} dt + \overbrace{F - \lambda S_T}^{\text{short position in the forward}},$$

$$P_T^c := \int_0^T \overbrace{c_t(p_0 + p_1 S_t)}^{\text{income from selling}} \underbrace{-\gamma c_t(S_t + \delta)}_{\text{sourcing costs}} \overbrace{- \frac{k_c}{2} v_t^2 - \frac{\ell_c}{2}(y_t - \sigma_c)^2}^{\text{costs of controls}} dt \underbrace{-F + \lambda S_T}_{\text{long in the forward}}.$$

- $k_p, \ell_p, \sigma_p, k_c, \ell_c, \sigma_c > 0$ and σ_p and σ_c nominal uncertainty in production (resp. consumption) in case of no effort.
- $c_t(p_0 + p_1 S_t)$ income from selling the quantity c_t at the retail price $(p_0 + p_1 S_t)$, with $p_0, p_1 > 0$.
- $\gamma c_t(S_t + \delta)$ sourcing cost of buying the quantity γc_t (to obtain c_t) at price S_t plus the transformation cost δ, with $\gamma, \delta > 0$ and $\gamma > p_1$ to ensure concavity of the obj. functional.
- The players exchange a forward contract of λ units of the commodity at a fixed price $F \in \mathbb{R}$.

Objective functionals & Equilibrium

The objective functionals of the maximization problems are:

$$J_p^{\lambda,F}(u,z;v,y) := \mathbb{E}\left[P_T^p\right] - \eta_p \int_0^T \mathbb{V}\left[\lambda S_t\right] dt \qquad \eta_p > 0, \tag{1}$$

$$J_c^{\lambda,F}(v,y;u,z) := \mathbb{E}\left[P_T^c\right] - \eta_c \int_0^T \mathbb{V}\left[\lambda S_t\right] dt, \qquad \eta_c > 0, \tag{2}$$

where $\mathbb{V}$ stands for the variance. We look for *Nash equilibria*:

Definition

We call the couple $\left((u^*,z^*)^\top,(v^*,y^*)^\top\right) \in \mathcal{A}\times\mathcal{A}$ a ***Nash equilibrium*** if

$$J_p^{\lambda,F}(u^*,z^*;v^*,y^*) \geq J_p^{\lambda,F}(u,z;v^*,y^*), \qquad \text{for all } (u,z)^\top \in \mathcal{A}, \tag{3}$$

$$J_c^{\lambda,F}(v^*,y^*;u^*,z^*) \geq J_c^{\lambda,F}(v,y;u^*,z^*), \qquad \text{for all } (v,y)^\top \in \mathcal{A}. \tag{4}$$

Risk aversion

The players have some risk-aversion in their financial position, which is modelled via an integrated variance penalization. Motivations:

- Utility functions (e.g. exp) would lead to nonlinear PDEs which are difficult to handle.
- Related papers on mean-var portfolio choice: [Zhou and Li (2000), Ismael and Pham (2019), Lefebvre et. al. (2020), Aïd et. al (2020)].
- This choice captures some separation in the production firm between a production unit and a trading unit.

N.B. Variance in the obj functionals: linear-quadratic McKean-Vlasov game formulation!

Theorem

Under technical assumptions, there exists a Nash equilibrium $((u^*, z^*)^\top, (v^*, y^*)^\top) \in \mathcal{A}^2$ *in the following feedback form*

$$u_t^* = \frac{2}{k_p}\Big[(K_p(t) + \pi_{11}(t))(q_t - \bar{q}_t) + \pi_{12}(t)(c_t - \bar{c}_t) + (\Lambda_p(t) + \widehat{\pi}_{11}(t))\bar{q}_t + \widehat{\pi}_{12}(t)\bar{c}_t + h_1(t)\Big], \qquad z^*(t) = \frac{\sigma_p \ell_p}{\ell_p - 2(K_p(t) + \pi_{11}(t))},$$

$$v_t^* = \frac{2}{k_c}\Big[(K_c(t) + \pi_{22}(t))(c_t - \bar{c}_t) + \pi_{21}(t)(q_t - \bar{q}_t) + (\Lambda_c(t) + \widehat{\pi}_{22}(t))\bar{c}_t + \widehat{\pi}_{21}(t)\bar{q}_t + h_2(t)\Big], \qquad y^*(t) = \frac{\sigma_c \ell_c}{\ell_c - 2(K_c(t) + \pi_{22}(t))}.$$

The equilibrium payoffs

$$J_p^*(\lambda, F) := J_p^{\lambda,F}(u^*, z^*; v^*, y^*) \quad \text{and} \quad J_c^*(\lambda, F) := J_c^{\lambda,F}(v^*, y^*; u^*, z^*)$$

have an explicit representation. Notation: $\bar{q}_t = \mathbb{E}[q_t]$ *and* $\bar{c}_t = \mathbb{E}[c_t]$.

A sketch of the proof

a. Compute the best response[2] (BR) maps of both players using a suitable verification theorem
   - The verification thm expresses the BR payoffs as expectations of suitable processes;
   - Ansatz on such processes as quadratic functions of the states;
   - The ansatz leads to a system of equations for the coefficients.

b. Check the system coming from the BR computations has a unique solution.

c. Get a Nash equilibrium as a fixed point of the BR maps.

d. Verify ∃! solution to the system in *c*.

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Indifference price

- The market is incomplete (two BMs and one traded risky asset): both players determine their forward prices using the **indifference pricing approach**.
- Key facts:

	$F_{\cdot}^{\lambda,*}$ solves UIP	Linearity
Consumer: $F_c^{\lambda,*}$	$J_c^*(\lambda, \mathbf{F}) = J_c^*(0,0)$	$J_c^*(\lambda, F) = J_c^*(\lambda, 0) - F$
Producer: $F_p^{\lambda,*}$	$J_p^*(\lambda, \mathbf{F}) = J_p^*(0,0)$	$J_p^*(\lambda, F) = J_p^*(\lambda, 0) + F$

- $F_c^{\lambda,*} = J_c^*(\lambda, 0) - J_c^*(0,0)$: the max the consumer is willing to pay.
- $F_p^{\lambda,*} = J_p^*(0,0) - J_p^*(\lambda, 0)$: the min the producer is willing to accept.

Agreement indifference price

As a consequence, trading is possible if and only if

$$F_p^{\lambda,*} \leq F_c^{\lambda,*}. \tag{5}$$

We look for the numerical value $\lambda > 0$ such that (5) holds as an equality.

Definition

Let λ^* be the number of units of the underlying so that $F_p^{\lambda^*,*} = F_c^{\lambda^*,*}$. The price

$$F_{\lambda^*}^* := F_p^{\lambda^*,*} = F_c^{\lambda^*,*} \qquad \text{and} \qquad f^{\lambda^*,*} := \frac{F_{\lambda^*}^*}{\lambda^*}.$$

are called **cumulative** agreement indifference price and **unitary** agreement indifference price, respectively.

The parameters

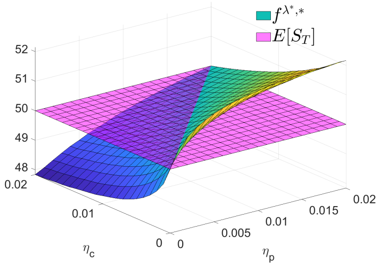
Numerical results obtained with Matlab:

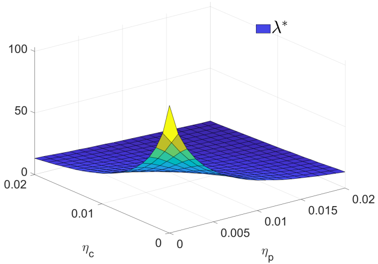
- Final time horizon: $T = 1$;
- States and spot price: $\rho_p = \gamma \rho_{\mathbf{c}} = 0.5$, $s_0 = 50$, $q_0 = c_0 = 100$;
- Objective functionals: $k_p = k_c = 5$, $\sigma_p = \sigma_c = 10$, $\ell_p = \ell_c = 5$, $\gamma = 1.2, \delta = 5$;
- Price of the final good: $p_0 = 2s_0 + \gamma\delta$, $p_1 = \gamma - 1$.

With this choice the players are *symmetric*, i.e.,

- same absolute effect on the price and the same cost of production/consumption,

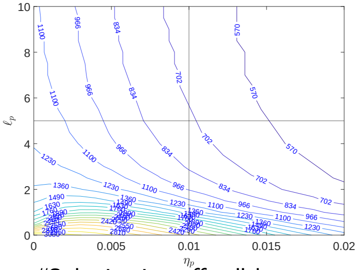
The effect of risk aversion on $f^{\lambda^*,*}$ and λ^*

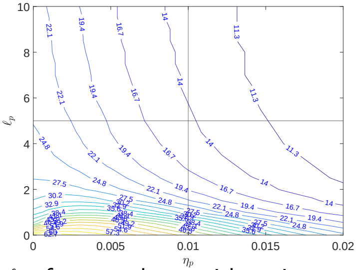




- $\eta_p \lesseqgtr \eta_c \Rightarrow$ (unitary) forward price $\lesseqgtr$ than the expected spot price $\rightsquigarrow$ risk premium consistent with intuition and hedging pressure theory: the most risk averse speculator obtains the most appropriate premium to enter the agreement; this result does not vary with level of vola control cost
- the higher the RA $\Rightarrow$ the lower the trading volume.

The joint effect of risk aversion and vola control cost on $F^*_{\lambda^*}$ and λ^* ($\eta_c = 0.01$, $\ell_c = 5$)





- "Substitution effect" between η_p and ℓ_p: for a producer with a given combination of η_p, ℓ_p, we find another producer *trading at the same agreement price* with a higher η_p and a lower ℓ_p.
- Same behaviour for the traded volume λ^*
- When ℓ_p is very high, the level lines are almost vertical $\Rightarrow$ further increase of ℓ_p has no effect

Conclusions

Mathematical results:

- Two-player nonzero-sum linear-quadratic stochastic differential game with McKean-Vlasov type objctive functionals.
- Existence of a Nash equilibrium with closed-form expressions for the corresponding strategies and the payoffs.

Economic insights:

- *Effect of risk aversion parameters on the forward price and traded volume*: the sign of risk premium is affected by the way players' risk aversions are ordered.
- *Joint effect of risk aversion parameters and volatility manipulation costs*: substitution effect between η_p and ℓ_p.
- *Cost of reducing production uncertainty as new determinant of the risk premium sign*.

Future developments

- Numerical results for different sets of parameters (non-symmetric cases);
- Nonlinear or exotic derivatives;
- Different forms of risk aversion.

Thanks for your attention!

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Thanks for your attention!

Short Bibliography I

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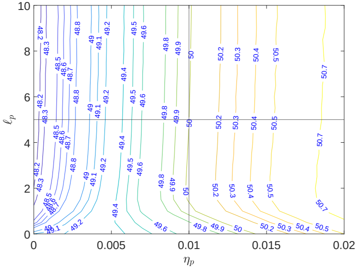
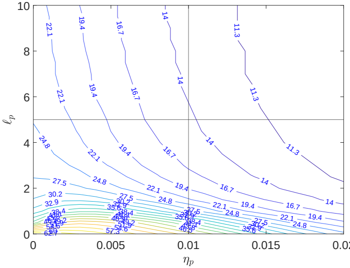
The joint effect of risk aversion and volatility control cost on $f^{\lambda^*,*}$ and λ^*





The volatility control cost has little effect on the per unit forward price compared to the risk-aversion parameter. When the volatility control costs are high, the producer has little alternative than asking for a premium to enter in forward agreement, and thus, the price is basically determined by his risk-aversion.