Portfolio Optimization for a Hilbert-Valued Stochastic Volatility Model with Jumps

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Power Markets

MOTIVATION FOR THE MODEL :

- The HJM framework for forward power prices can exhibit a truly infinite-dimensional dynamics...
- But we need more flexibility for modeling volatility (Samuelson effect, etc.) : BNS can be a suitable setting...
- and we want to allow jumps to appear in the asset dynamics as well (not only in the volatility).

WE INVESTIGATE THE OPTIMAL PORTFOLIO PROBLEM IN THIS SETTING

REFERENCES I

SOME BASIC LITERATURE :

- Barndorff-Nielsen, O. E., Shephard, N. (2001) : Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in economics. *J.R. Statistic. Soc. B*, 63 (2), 167-241.
- Peszat, S., Zabczyk, J. (2007) : *Stochastic Partial Differential Equations with Lévy Noise*. Cambridge University Press, Cambridge.
- Benth, F. E., Ruediger, B. and Suss, A. (2018) : Ornstein-Uhlenbeck Processes in Hilbert Space with non-Gaussian Stochastic Volatility, *Stochastic Processes and Applications*, 13(4), 543-577.
- Benth, F. E., Sgarra, C. (2022) : A Barndorff-Nielsen and Shephard model with leverage in Hilbert space for commodity forward markets. Submitted. Available on SSRN.

REFERENCES II

SOME MORE LITERATURE :

- Benth, F. E., Lempa, J. (2014) : Optimal portfolios in commodity futures markets, *Finance Stoch.*, 18, 407-430.
- Święch, A., Zabczyk, J. (2013) : Uniqueness for Integro-PDE in Hilbert Spaces. *Potential Analysis*, 38, 233-259.
- Święch, A., Zabczyk, J. (2016) : Integro-PDE in Hilbert Spaces : Existence of Viscosity Solutions. *Potential Analysis*, 45, 703-736.

H-valued BNS-SV Model

Let X be a stochastic process with values in the Hilbert space H satisfying the following stochastic differential equation : Let $X = (X(t))_{t\geq 0}$ be a stochastic process taking

values in the Hilbert space H satisfying the following stochastic differential equation :

$$\begin{aligned} dX(t) &= (\mathcal{A}\, X(t) + b(t))\, dt + \mathcal{Y}^{1/2}(t)\, dB(t) - \mathbf{\Gamma}\, d(t), \\ X(0) &= X_0, \end{aligned} \tag{1}$$

with stochastic volatility

$$\begin{aligned} d\mathcal{Y}(t) &= \boldsymbol{C}\, \mathcal{Y}(t)\, dt + d(t), \\ \mathcal{Y}(0) &= \mathcal{Y}_0. \end{aligned} \tag{2}$$

Here X_0 (resp. Y_0) is a fixed element of H (resp. $\mathcal{H}$), while the coefficients $\mathcal{A}$, $\boldsymbol{C}$, β, b

satisfy the following standing assumptions.

1. The operator $\mathcal{A}\colon D(\mathcal{A}) \subset H \to H$ is a linear, densely defined, operator and is the infinitesimal generator of a strongly continuous semigroup on H, denoted by $\{\mathcal{S}(t), t \geq 0\}$.
2. $\boldsymbol{C}$ is a bounded linear operator on $\mathcal{H}$, namely $\boldsymbol{C} \in \mathcal{L}(\mathcal{H})$, therefore $\boldsymbol{C}$ is the infinitesimal generator of a uniformly continuous semigroup on $\mathcal{H}$, denoted by $\{\boldsymbol{S}(t), t \geq 0\}$.
3. $b\colon [0, +\infty) \to H$ is Lipschitz continuous, namely

$$|b(t) - b(t')|_H \ \leq \ c\,|t - t'|, \qquad \forall\, t, t' \geq 0, \tag{3}$$

   for some constant $c \geq 0$.
4. $\boldsymbol{\Gamma}$ is a bounded linear operator from $\mathcal{H}$ into H, namely $\boldsymbol{\Gamma} \in \mathcal{L}(\mathcal{H}; H)$. We also assume that $\boldsymbol{\Gamma}$ is preserving the non-decreasing path property.

Moreover $X_0 \in H$ and $\mathcal{Y}$ is a $\mathcal{H}$-valued stochastic process satisfying the following stochastic differential equation :

$$d\mathcal{Y}(t) = \mathfrak{C}\mathcal{Y}(t)dt + d\mathcal{L}, \quad \mathcal{Y}(0) = \mathcal{Y}_0, \tag{4}$$

where we assume that $\mathcal{Y}(0)$ is self-adjoint, non-negative definite and $\mathcal{L}$ is an $\mathcal{H}$-valued Lévy process with non-decreasing paths. ρ is a linear, positive and bounded operator acting on $\mathcal{L}$, mapping L_{HS} into H.

$\mathcal{A}$ is a linear operator on H, possibly unbounded, densely defined, generating a C_0-semigroup $\mathcal{S}$. In Benth, Ruediger and Suess (SPA 2018) a detailed investigation on the operator $\mathbb{C}$ and the conditions granting the positivity of the process $\mathcal{Y}$ are provided.

BNS with Leverage

We have the following useful Lemma concerning the leverage term :

Lemma

Assume $\mathbf{\Gamma} \in L(\mathcal{H}, H)$ and $\mathcal{L}$ is a Lévy process taking values in $\mathcal{H}$. Then, $\mathbf{\Gamma}\mathcal{L}(t)$ is an H-valued Lévy process with Lévy-Kintchine representation

$$\mathbb{E}\left[\exp(\mathrm{i}(\mathbf{\Gamma}\mathcal{L}(1), h)_H)\right] = \exp(\Psi_{\mathcal{L}}(\mathbf{\Gamma}^* h))$$

for all $h \in H$, where $\Psi_{\mathcal{L}}(\mathcal{T}), \mathcal{T} \in \mathcal{H}$ is the characteristic exponent of $\mathcal{L}$.

Sketch of the Proof : From Peszat and Zabczyk, $\mathbf{\Gamma}\mathcal{L}$ is a Lévy process in H. Moreover, for any $h \in H$ we find that $(\mathbf{\Gamma}\mathcal{L}(1), h)_H = \langle \mathcal{L}(1), \mathbf{\Gamma}^* h \rangle_{\mathcal{H}}$, and the Lévy-Kintchine representation follows.

We have the following mild solution :

$$X(t) = \mathcal{S}(t)X_0 + \int_0^t \mathcal{S}(t-u)R(u)du + \int_0^t \mathcal{S}(t-u)\mathcal{Y}^{1/2}(u)dB(u) + \int_0^t \mathcal{S}(t-u)\mathbf{\Gamma}\, d\mathcal{L}(u). \tag{5}$$

Notice that by assumption on R and $\mathcal{S}$ being a C_0-semigroup, the first integral above is well-defined. The last integral is also well-defined, as $\mathbf{\Gamma}\mathcal{L}$ is a Lévy process in H.

In the present section we consider a specific Hilbert space H, namely we will take $H = H_\alpha$, with H_α being the Filipović space. We also denote $\mathcal{H}$ by $\mathcal{H}_\alpha$ and fix the operator $\mathcal{A}$, which will be given by ∂_x.

We fix some notations. Let $\mathbb{R}_+ := [0, +\infty)$ and let $AC(\mathbb{R}_+)$ denote the set of absolutely continuous functions $h\colon \mathbb{R}_+ \to \mathbb{R}$. We also denote by $L(\mathbb{R}_+)$ (resp. $L_{\text{loc}}(\mathbb{R}_+)$) the set of functions $h\colon \mathbb{R}_+ \to \mathbb{R}$ which are integrable (resp. locally integrable) with respect to the Lebesgue measure on $\mathbb{R}_+$.

Moreover, we recall that given $h \in AC(\mathbb{R}_+)$, the weak derivative $h' \in L^1_{\text{loc}}(\mathbb{R}_+)$ of h, if it exists, is uniquely specified by the property :

$$\int_{\mathbb{R}_+} h(x)\, \varphi'(x)\, dx \; = \; - \int_{\mathbb{R}_+} h'(x)\, \varphi(x)\, dx,$$

for every $\varphi \in C^1_c((0, \infty))$, with $C^1_c((0, +\infty))$ being the set of C^1-functions having compact support in $(0, \infty)$.

Standing Assumption : Let $\alpha\colon [0,+\infty) \to [1,+\infty)$ be a fixed non-decreasing continuous function satisfying $\alpha(0) = 1$.

Definition

For every $h \in AC(\mathbb{R}_+)$, we write

$$\|h\|_{\alpha^2} := |h(0)|^2 + \int_0^\infty |h'(x)|^2 \, \alpha(x) \, dx$$

and define the Filipović space

$$H_\alpha := \{h \in AC(\mathbb{R}_+)\colon \|h\|_\alpha < \infty\}.$$

Remark : Notice that H_α turns out to be a separable Hilbert space, with scalar product

$$(h, g)_\alpha \ := \ h(0)\, g(0) + \int_0^\infty h'(x)\, g'(x)\, \alpha(x)\, dx.$$

We also remark that, given a fixed $x \in \mathbb{R}_+$, the point evaluation $\delta_x\colon\ H_\alpha \to \mathbb{R}$, defined as $h \mapsto \delta_x(h) := h(x)$, is a continuous linear functional on H_α. In the present section we suppose that the Hilbert space H is given by H_α.

Proposition : Let $\partial_x : D_\alpha \to H_\alpha$ be defined as $h \mapsto h'$, with $D_\alpha := \{h \in H_\alpha : h' \in H_\alpha\}$. Then, ∂_x is the infinitesimal generator of the strongly continuous semigroup on H_α, denoted by $\{\mathcal{S}_{hift}(t), t \geq 0\}$, corresponding to the right shift operator :

$$(\mathcal{S}_{hift}(t)h)(x) \; = \; h(x+t), \qquad \forall\, x \in \mathbb{R}_+,\, h \in H_\alpha,$$

for every $t \geq 0$. The semigroup $\{\mathcal{S}_{hift}(t), t \geq 0\}$ is quasi-contractive, namely there exists $\beta_0 > 0$ such that

$$\|\mathcal{S}_{hift}(t)\|_{\mathcal{L}(H_\alpha)} \; \leq \; e^{\beta_0 t}, \tag{6}$$

for every $t \geq 0$.

Proof : see Filipovic (2001)[Theorem 5.1.1, Remark 5.1.1] and Benth and Kruehner (2014)[Theorem 3.4, Lemma 3.5].

Standing Assumption : The Hilbert space H is equal to H_α and, consequently, we denote $\mathcal{H}$ by $\mathcal{H}_\alpha$. Moreover, the operator $\mathcal{A}$ is given by ∂_x (with domain $D(\mathcal{A}) = D_\alpha$).

In the present section, we provide the dynamics of the wealth generated by trading on future contracts. A futures contract is a derivative security written on the futures price, which is denoted by $f(t)$. Notice that, since interest rates are assumed to be constant, in the present context forward and futures prices can be identified. Consider X, mild solution to our equation with $H = H_\alpha$, $\mathcal{H} = \mathcal{H}_\alpha$, $\mathcal{A} = \partial_x$. Then, the process X can be

thought as the dynamics of the forward curve, namely

$$f(t,x) \ := \ \delta_x(X(t)) \ = \ X(t)(x),$$

where $f(t,x) = F(t, t+x)$ (also denoted by $f(t)(x)$) and $(F(t,T))_{t\in[0,T]}$ is the forward price dynamics of a contract delivering at time T.

As a consequence, we have that f solves the following stochastic differential equation :

$$\begin{aligned} f(t) \;=\; & \mathcal{S}(t)\, f_0 + \int_0^t \mathcal{S}(t-u)\, b(u)\, du + \int_0^t \mathcal{S}(t-u)\, \mathcal{Y}^{1/2}(u)\, dB(u) \\ & - \int_0^t \mathcal{S}(t-u)\, \boldsymbol{\Gamma}\, d(u) \end{aligned} \tag{7}$$

or, equivalently,

$$\begin{aligned} f(t,x) \;=\; & \mathcal{S}(t)\, f_0(x) + \delta_x \int_0^t \mathcal{S}(t-u)\, b(u)\, du + \delta_x \int_0^t \mathcal{S}(t-u)\, \mathcal{Y}^{1/2}(u)\, dB(u) \\ & - \delta_x \int_0^t \mathcal{S}(t-u)\, \boldsymbol{\Gamma}\, d(u), \end{aligned} \tag{8}$$

with $f_0(x) := \delta_x(X_0) = X_0(x)$.

For every $t \geq 0$, let $(\mathcal{G}_s^t)_{s \geq t}$ be the standard augmentation of the filtration generated by $(B(s) - B(t))_{s \geq t}$ and $(L(s) - L(t))_{s \geq t}$. Let also $M > 0$ and denote by $B^M := \{p \in H_\alpha^* \colon |p|_{H_\alpha^*} \leq M\}$ a fixed ball in the dual space H_α^*.

Then, for every $t \geq 0$, we denote by Π_t^M the set of futures portfolios on the time interval $[t, T]$, namely the set of all $(\mathcal{G}_s^t)_{s \geq t}$-predictable processes $\pi \colon [t, T] \times \Omega \to H_\alpha^*$ taking values in B^M (in other words, Π_t^M is the set of all $(\mathcal{G}_s^t)_{s \geq t}$-predictable processes taking values in the dual space H_α^*, which are uniformly bounded by M).

Given an initial time $t \in [0, T]$, an initial wealth $w \in \mathbb{R}$, an initial volatility $\mathcal{Z} \in \mathcal{H}_\alpha$, and a futures portfolio $\pi \in \Pi_t^M$, we define the wealth generated by such a portfolio as follows :

$$\begin{aligned} dW^{t,w,\mathcal{Z},\pi}(s) &= \left[rW^{t,w,\mathcal{Z},\pi}(s) + \langle \pi(s), b(s)\rangle\right] ds + \left\langle \pi(s), (\mathcal{Y}^{t,\mathcal{Z}}(s))^{1/2}\, dB(s)\right\rangle \\ &\quad - \left\langle \pi(s), \mathbf{\Gamma}\, dL(s)\right\rangle, \qquad t \leq s \leq T, \\ W^{t,w,\mathcal{Z},\pi}(t) &= w, \end{aligned} \tag{9}$$

with $\mathcal{Y}^{t,\mathcal{Z}}$ mild solution to the following equation :

$$\begin{aligned} \mathcal{Y}^{t,\mathcal{Z}}(s) &= \boldsymbol{C}\,\mathcal{Y}^{t,\mathcal{Z}}(s)\, ds + dL(s), \qquad t \leq s \leq T, \\ \mathcal{Y}^{t,\mathcal{Z}}(t) &= \mathcal{Z}, \end{aligned} \tag{10}$$

where $r > 0$ denotes the risk-free rate of return, while $\langle \cdot, \cdot \rangle \colon H_\alpha^* \times H_\alpha \to \mathbb{R}$ is the natural bilinear pairing between H_α and its dual H_α^*.

Proposition : Let $0 \leq t \leq t_1 < T$ and $\pi \in \Pi_t^M$. Let also $\xi \colon \Omega \to \mathbb{R}$ and $\Xi \colon \Omega \to \mathcal{H}_\alpha$ be $\mathcal{G}_{t_1}^t$-measurable and such that $\mathbb{E}|\xi|^2 + \mathbb{E}|\Xi|^2_{\mathcal{H}_\alpha} < \infty$. Then, there exists a unique (up to $\mathbb{P}$-indistinguishability) mild solution $(W^{t_1,\xi,\Xi,\pi}(s), \mathcal{Y}^{t_1,\Xi}(s))_{s\in[t_1,T]}$ to our system explicitly given by the following formulae :

$$\begin{aligned} W^{t_1,\xi,\Xi,\pi}(s) &= \xi + \int_{t_1}^{s} e^{r(s-u)} \langle \pi(u), b(u) \rangle \, du + \int_{t_1}^{s} e^{r(s-u)} \big\langle \pi(u), (\mathcal{Y}^{t_1,\Xi}(u))^{1/2} \, dB(u) \big\rangle \\ &\quad - \int_{t_1}^{s} e^{r(s-u)} \big\langle \pi(u), \boldsymbol{\Gamma} \, dL(u) \big\rangle, \qquad t_1 \leq s \leq T, \\ \mathcal{Y}^{t_1,\Xi}(s) &= \boldsymbol{S}(s - t_1)\, \Xi + \int_{t_1}^{s} \boldsymbol{S}(s-u) \, dL(u), \qquad t_1 \leq s \leq T, \end{aligned}$$

where $\{\boldsymbol{S}(u), u \geq 0\}$ is the uniformly continuous semigroup with infinitesimal generator $\boldsymbol{C}$.

Moreover, it holds that

$$\mathbb{E}\left[\sup_{t_1\leq s\leq T}\left(\left|W^{t_1,\xi,\Xi,\pi}(s)\right|^2+\left|\mathcal{Y}^{t_1,\Xi}(s)\right|^2_{\mathcal{H}_\alpha}\right)\right]\ \leq\ C\left(1+\mathbb{E}|\xi|^2+\mathbb{E}|\Xi|^2_{\mathcal{H}_\alpha}\right),$$

for some positive constant C, not depending on t, t_1, ξ, Ξ, π. Proof : See Theorem 3.4 in Swiech and Zabczyk (2013).

The optimal control problem consists in finding a futures portfolio maximizing the expected utility from terminal wealth :

$$V(t,w,\mathcal{Z})\ =\ \sup_{\pi\in\Pi_t^M}\mathbb{E}\left[\mathcal{U}(W^{t,w,\mathcal{Z},\pi}(T))\right], \tag{11}$$

for every $(t, w, \mathcal{Z}) \in [0, T]\times\mathbb{R}\times\mathcal{H}_\alpha$.

Proposition : The value function V satisfies the following properties :

$$
\begin{aligned}
|V(t,w,\mathcal{Z}) - V(t,w',\mathcal{Z}')| &\le \sigma\big(|w-w'| + |\mathcal{Z}-\mathcal{Z}'|_{\mathcal{H}_\alpha}\big), & t \in [0,T],\ (w,\mathcal{Z}) \in \mathbb{R}\times\mathcal{H}_\alpha,\\
|V(t,w,\mathcal{Z}) - V(s,w,\mathcal{Z})| &\le \sigma_R(|t-s|), & t,s \in [0,T],\ |w|, |\mathcal{Z}|_{\mathcal{H}_\alpha} \le R,\\
|V(t,w,\mathcal{Z})| &\le C\left(1 + |w| + |\mathcal{Z}|_{\mathcal{H}_\alpha}\right), & (t,w,\mathcal{Z}) \in [0,T]\times\mathbb{R}\times\mathcal{H}_\alpha
\end{aligned}
$$

for some positive constant C, where, for every $R > 0$, σ_R is a modulus of continuity,

namely a continuous subadditive function on $[0,\infty)$ such that $\sigma_R(0) = 0$ and $\sigma_R(a) > 0$, whenever $a > 0$; finally, σ is also a modulus of continuity.

Proof : The claim follows from Lemma 4.1 and Lemma 4.3 in Swiech and Zabczyk (2013) and also from our previous Remark on continuity.

THEOREM [Dynamic Programming Principle] :

Let $0 \leq t \leq t_1 \leq T$, $w \in \mathbb{R}$, $\mathcal{Z} \in \mathcal{H}_\alpha$, and $\pi \in \Pi_t^M$. Then

$$V(t, w, \mathcal{Z}) \; = \; \sup_{\pi \in \Pi_t^M} \mathbb{E}\Big[V\big(t_1, W^{t,w,\mathcal{Z},\pi}(t_1), \mathcal{Y}^{t,\mathcal{Z}}(t_1)\big)\Big].$$

Proof : The claim follows from Theorem 3.14 and Theorem 4.2 in Swiech and Zabczyk (2016).

The Hamilton-Jacobi-Bellman equation associated with such an optimization problem turns out to be the following integro-PDE :

$$
\begin{cases}
\partial_t V(t,w,\mathcal{Z}) + \sup_{p\in H^*_\alpha} \Big\{ \big(r\,w + \langle p, b(t)\rangle - \langle p, \mathbf{\Gamma}\mathcal{D}\rangle\big)\partial_w V(t,w,\mathcal{Z}) \\
+ \big(\boldsymbol{C}\mathcal{Z} + \mathcal{D}, \partial_{\mathcal{Z}} V(t,w,\mathcal{Z})\big)_{\mathcal{H}_\alpha} + \int_{\mathcal{H}_\alpha} \big\{ V(t, w - \langle p, \mathbf{\Gamma}\mathcal{R}\rangle, \mathcal{Z}+\mathcal{R}) - V(t,w,\mathcal{Z}) \\
- 1_{\{|\mathcal{R}|_{\mathcal{H}_\alpha}<1\}}\big((\mathcal{R}, \partial_{\mathcal{Z}} V(t,w,\mathcal{Z}))_{\mathcal{H}_\alpha} - \langle p, \mathbf{\Gamma}\mathcal{R}\rangle \partial_w V(t,w,\mathcal{Z})\big)\big\}\boldsymbol{\nu}(d\mathcal{Z}) \\
+ \frac{1}{2}\big|\langle p, \mathcal{Z}^{1/2}\mathcal{Q}^{1/2}\cdot\rangle\big|_2^2 \partial^2_{ww} V(t,w,\mathcal{Z}) + \langle p, \mathbf{\Gamma}\boldsymbol{Q}^c \partial^2_{w\mathcal{Z}} V(t,w,\mathcal{Z})\rangle \\
+ \frac{1}{2}\mathrm{Tr}\big[\boldsymbol{Q}^c \partial^2_{\mathcal{Z}\mathcal{Z}} V(t,w,\mathcal{Z})\big] \Big\} \;=\; 0, & (t,w,\mathcal{Z}) \in [0,T)\times\mathbb{R}\times\mathcal{H}_\alpha, \\
V(T,w,\mathcal{Z}) \;=\; \mathcal{U}(w), & (w,\mathcal{Z}) \in \mathbb{R}\times\mathcal{H}_\alpha,
\end{cases}
\tag{12}
$$

where, for every fixed $p \in H_\alpha^*$, $\langle p, \mathcal{Z}^{1/2}\mathcal{Q}^{1/2}\cdot\rangle$ is the functional from H_α into $\mathbb{R}$ such that $h \mapsto \langle p, \mathcal{Z}^{1/2}\mathcal{Q}^{1/2}h\rangle$, and

$$\left|\langle p, \mathcal{Z}^{1/2}\mathcal{Q}^{1/2}\cdot\rangle\right|_2 := \left(\sum_{n\in\mathbb{N}} \left|\langle p, \mathcal{Z}^{1/2}\mathcal{Q}^{1/2}e_n\rangle\right|^2\right)^{1/2},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H_α.

Definition : We say that $\psi\colon [0, T] \times \mathbb{R} \times \mathcal{H}_\alpha \to \mathbb{R}$ is a test function if

$$\psi(t, w, \mathcal{Z}) \;=\; \varphi(t, w, \mathcal{Z}) + \delta(t, w, \mathcal{Z})\, h\Big(\sqrt{|w|^2 + |\mathcal{Z}|^2_{\mathcal{H}_\alpha}}\Big), \tag{13}$$

where :

i) φ is bounded and $\partial_t\varphi$, $\partial_w\varphi$, $\partial_{\mathcal{Z}}\varphi$, $\partial^2_{ww}\varphi$, $\partial^2_{w\mathcal{Z}}\varphi$, $\partial^2_{\mathcal{Z}\mathcal{Z}}\varphi$ are uniformly continuous on $(\varepsilon, T-\varepsilon) \times \mathbb{R} \times \mathcal{H}_\alpha$, for every $\varepsilon > 0$.

ii) δ is non-negative and bounded, moreover $\partial_t\delta$, $\partial_w\delta$, $\partial_{\mathcal{Z}}\delta$, $\partial^2_{ww}\delta$, $\partial^2_{w\mathcal{Z}}\delta$, $\partial^2_{\mathcal{Z}\mathcal{Z}}\delta$ are uniformly continuous on $(\varepsilon, T-\varepsilon) \times \mathbb{R} \times \mathcal{H}_\alpha$, for every $\varepsilon > 0$.

iii) h is even and bounded, h' and h'' are uniformly continuous on $\mathbb{R}$, $h'(a) \geq 0$, for every $a > 0$.

Definition : Let $U\colon [0,T] \times \mathbb{R} \times \mathcal{H}_\alpha \to \mathbb{R}$ be a function.

- U is said to be a viscosity subsolution of HJB if it is upper-semicontinuous,

$$U(T, w, \mathcal{Z}) \ \leq \ \mathcal{U}(w), \qquad \text{for every } (w, \mathcal{Z}) \in \mathbb{R} \times \mathcal{H}_\alpha$$

  and whenever $U - \psi$ has a global maximum at a point $(t, w, \mathcal{Z}) \in [0, T) \times \mathbb{R} \times \mathcal{H}_\alpha$ for a test function ψ, then HJB holds with V and $=$ replaced respectively by ψ and $\geq$.

- U is said to be a viscosity supersolution of HJB if it is lower-semicontinuous,

$$U(T, w, \mathcal{Z}) \ \geq \ \mathcal{U}(w), \qquad \text{for every } (w, \mathcal{Z}) \in \mathbb{R} \times \mathcal{H}_\alpha$$

  and whenever $U + \psi$ has a global minimum at a point $(t, w, \mathcal{Z}) \in [0, T) \times \mathbb{R} \times \mathcal{H}_\alpha$ for a test function ψ, then HJB holds with V and $=$ replaced respectively by $-\psi$ and $\leq$.

- U is said to be viscosity solution of if it is continuous and it is both a viscosity subsolution and a viscosity supersolution.

THEOREM :

The value function V is a viscosity solution of HJB. If in addition V is bounded and uniformly continuous on $[0, T] \times \mathbb{R} \times \mathcal{H}_\alpha$, then it is the unique viscosity solution of HJB in the class of bounded and uniformly continuous functions on $[0, T] \times \mathbb{R} \times \mathcal{H}_\alpha$.

Proof : The existence part follows from Theorem 5.4 in Swiech and Zabczyk (2016), while the uniqueness part is a consequence of Theorem 6.2 in Swiech and Zabczyk (2013).

Theorem : Suppose that there exist $\hat{V}\colon [0, T] \times \mathbb{R} \times \mathcal{H}_\alpha \to \mathbb{R}$ and $\hat{\mathrm{p}}\colon [0, T] \times \mathbb{R} \times \mathcal{H}_\alpha \to H^*_\alpha$ satisfying the following conditions.

- i) $\hat{V} \in C^{1,2,2}([0, T] \times \mathbb{R} \times \mathcal{H}_\alpha)$ and $\hat{V}$ is a classical solution of HJB.
- ii) There exists a positive constant $\hat{C}$ such that

$$|\hat{V}(t, w, \mathcal{Z})| \ \leq \ \hat{C}\left(1 + |w| + |\mathcal{Z}|_{\mathcal{H}_\alpha}\right),$$

  for all $(t, w, \mathcal{Z}) \in [0, T] \times \mathbb{R} \times \mathcal{H}_\alpha$.
- iii) $\hat{\mathrm{p}}\colon [0, T] \times \mathbb{R} \times \mathcal{H}_\alpha \to H^*_\alpha$ is a Borelian function such that, for every fixed $(t, w, \mathcal{Z})$, the supremum appearing in HJB, with V replaced by $\hat{V}$, is attained at $p = \hat{\mathrm{p}}(t, w, \mathcal{Z})$.
- iv) For every $(t, w, \mathcal{Z}) \in [0, T] \times \mathbb{R} \times \mathcal{H}_\alpha$, there exists a $(\mathcal{G}^t_s)_{s\geq t}$-adapted and càdlàg process $(\hat{W}^{t,w,\mathcal{Z}}(s))_{s\in[t,T]}$ solution to the wealth equation controlled by $\hat{\mathrm{p}}$, namely

$$
\begin{aligned}
\hat{W}^{t,w,\mathcal{Z}}(s) \;=\; & w + \int_t^s e^{r(s-u)} \langle \hat{\mathrm{p}}(u, \hat{W}^{t,w,\mathcal{Z}}(u), \mathcal{Y}^{t,\mathcal{Z}}(u)), b(u)\rangle \, du \\
& + \int_t^s e^{r(s-u)} \big\langle \hat{\mathrm{p}}(u, \hat{W}^{t,w,\mathcal{Z}}(u), \mathcal{Y}^{t,\mathcal{Z}}(u)), (\mathcal{Y}^{t,\mathcal{Z}}(u))^{1/2} \, dB(u) \big\rangle \\
& - \int_t^s e^{r(s-u)} \big\langle \hat{\mathrm{p}}(u, \hat{W}^{t,w,\mathcal{Z}}(u), \mathcal{Y}^{t,\mathcal{Z}}(u)), \boldsymbol{\Gamma} \, dL(u) \big\rangle, \qquad t \le s \le T,
\end{aligned}
$$

with $(\mathcal{Y}^{t,\mathcal{Z}}(s))_{s\in[t,T]}$ such that

$$
\mathcal{Y}^{t,\mathcal{Z}}(s) \;=\; \boldsymbol{S}(s-t)\,\Xi + \int_t^s \boldsymbol{S}(s-u)\, d(u), \qquad t \le s \le T,
$$

Moreover, the stochastic process $\hat{\pi}$, defined as

$$
\hat{\pi}(s) \;=\; \hat{\mathrm{p}}(s, \hat{W}^{t,w,\mathcal{Z}}(s), \mathcal{Y}^{t,\mathcal{Z}}(s)), \qquad s \in [t, T],
$$

belongs to Π_t^M.
Then, it holds that $\hat{V} \equiv V$ and $\hat{\pi}$ is an optimal control.
Proof : The claim follows directly from an application of Itô's formula between $s = t$ and $s = T$ to $s \mapsto V(s, W^{t,w,\mathcal{Z},\pi}(s), \mathcal{Y}^{t,\mathcal{Z}}(s))$, with $\pi \in \Pi_t^M$, and to $s \mapsto V(s, \hat{W}^{t,w,\mathcal{Z}}(s), \mathcal{Y}^{t,\mathcal{Z}}(s))$.

The NO-LEVERAGE case

Now we want to show that an explicit solution to the optimal portfolio problem in the present setting in available for the NO-LEVERAGE case ($\Gamma = 0$).

The HJB equation in this case can be written as follows :

$$\begin{cases} \partial_t V(t,w,\mathcal{Z}) + \sup_{p \in H^*_\alpha} \left\{ \left(r\,w + \langle p, b(t) \rangle + \dfrac{1}{2} \left| \langle p, \mathcal{Z}^{1/2} \mathcal{Q}^{1/2} \cdot \rangle \right|_2^2 \partial^2_{ww} V(t,w,\mathcal{Z}) \right\} \\ + \left(\boldsymbol{C} \mathcal{Z} + \mathcal{D}, \partial_{\mathcal{Z}} V(t,w,\mathcal{Z}) \right)_{\mathcal{H}_\alpha} + \dfrac{1}{2} \mathrm{Tr} \left[\boldsymbol{Q}^c \partial^2_{\mathcal{Z}\mathcal{Z}} V(t,w,\mathcal{Z}) \right] + \\ + \int_{\mathcal{H}_\alpha} \left\{ V(t,w,\mathcal{Z}+\mathcal{R}) - V(t,w,\mathcal{Z}) - 1_{\{|\mathcal{R}|_{\mathcal{H}_\alpha} < 1\}} ((\mathcal{R}, \partial_{\mathcal{Z}} V(t,w,\mathcal{Z}))_{\mathcal{H}_\alpha} - \right\} \boldsymbol{\nu}(d\mathcal{R}) = \\ (t,w,\mathcal{Z}) \in [0,T) \times \mathbb{R} \times \mathcal{H}_\alpha, V(T,w,\mathcal{Z}) = \mathcal{U}(w), (w,\mathcal{Z}) \in \mathbb{R} \times \mathcal{H}_\alpha. \end{cases} \tag{14}$$

An explicit solution

We assume that the utility function is of power type, i.e. $\mathcal{U}(x) = \gamma^{-1}x^{\gamma}$ with $\gamma \in (0, 1)$. We guess a solution $V(t, w, \mathcal{Z})$ of the form $V(t, w, \mathcal{Z}) = \gamma^{-1}w^{\gamma}h(t, \mathcal{Z})$. The HJB equation becomes then :

$$\begin{cases} \partial_t h(t,\mathcal{Z}) + \gamma\Pi(\mathcal{Z})h(t,\mathcal{Z}) + \big(\boldsymbol{C}\mathcal{Z} + \mathcal{D}, \partial_{\mathcal{Z}} h(t,\mathcal{Z})\big)_{\mathcal{H}_\alpha} + \dfrac{1}{2}\mathrm{Tr}\big[\boldsymbol{Q}^c \partial^2_{\mathcal{Z}\mathcal{Z}} h(t,\mathcal{Z})\big] + \\ + \displaystyle\int_{\mathcal{H}_\alpha} \{h(t,\mathcal{Z}+\mathcal{R}) - h(t,\mathcal{Z}) - 1_{\{|\mathcal{R}|_{\mathcal{H}_\alpha}<1\}}((\mathcal{R}, \partial_{\mathcal{Z}} h(t,\mathcal{Z}))_{\mathcal{H}_\alpha} - \}\boldsymbol{\nu}(d\mathcal{R}) \;=\; 0, \\ (t,\mathcal{Z}) \in [0,T) \times \mathbb{R} \times \mathcal{H}_\alpha, \;\; h(T,\mathcal{Z}) \;= 1, \mathcal{Z} \in \mathcal{H}_\alpha, \end{cases} \tag{15}$$

where $\Pi(\mathcal{Z})$ is defined by :

$$\Pi(\mathcal{Z}) := \sup_{p \in H_\alpha^*} \left\{ \left(r + \langle p, b(t) \rangle + \frac{1}{2} \left| \langle p, \mathcal{Z}^{1/2} \mathcal{Q}^{1/2} \cdot \rangle \right|_2^2 (1 - \gamma) \right\}. \tag{16}$$

In order to get a more explicit representation for $\Pi(\mathcal{Z})$, we need to find the optimum in the previous expression. A first order condition provides the following relation :

$$b(t) + (1 - \gamma) \sum_{n \in \mathbb{N}} \mathcal{Z}^{1/2} \mathcal{Q}^{1/2} e_n \langle p, \mathcal{Z}^{1/2} \mathcal{Q}^{1/2} \cdot \rangle = 0. \tag{17}$$

by projecting both p and b on the orthonormal basis e_n we can write :

$$p(t) = \sum_{n \in \mathbb{N}} p_n(t) e_n \tag{18}$$

$$b(t) = \sum_{n \in \mathbb{N}} b_n(t) e_n, \tag{19}$$

in such a way that the first-order condition can be rewritten as follows :

$$b_n(t) e_n - (1 - \gamma) p_n(t) \mathcal{Z}^{1/2} \mathcal{Q}^{1/2} e_n \langle \mathcal{Z}^{1/2} \mathcal{Q}^{1/2} e_n, e_n \rangle = 0, \tag{20}$$

and these equations should hold for every $n \in \mathbb{N}$.

By writing

$$\mathcal{Z}^{1/2}\mathcal{Q}^{1/2}e_n = \lambda_n e_n, \tag{21}$$

we get :

$$b_n(t)e_n - (1-\gamma)p_n(t)\lambda_n^2 e_n = 0, \tag{22}$$

for every $n \in \mathbb{N}$.
By multiplying both sides on the left by e_n^*, we finally obtain :

$$p_n(t) = b_n(t)\frac{1}{(1-\gamma)\lambda_n^2} \tag{23}$$

Remark : discussion on the spectral properties of Q, $\mathcal{Z}$, $\mathcal{Z}^{1/2}\mathcal{Q}^{1/2}$.

Let's define the function $g(t, \mathcal{Z})$ as follows :

$$g(t, \mathcal{Z}) = \mathbb{E}\big[\exp\big(\int_0^t \gamma \Pi(\mathcal{Z}(u)) du\big)\big], \tag{24}$$

We assume an integrability condition of the following kind :

$$\int_{\|\mathcal{Z}\|_{\mathcal{H}} \geq 1} \exp\big(\langle \mathcal{Z}, \Theta \rangle_{\mathcal{H}}\big)\, \nu(d\mathcal{Z}) < \infty, \tag{25}$$

for $\Theta \in \mathcal{H}$.

Lemma : Assume Condition given before holds, then, for some positive constant k :

$$g(t, \mathcal{Z}) \leq \exp\left[kt + (B + \boldsymbol{C})^{-1}\mathcal{Z}\right], \tag{26}$$

where $B := \sup \Pi(\mathcal{Z})$.

Lemma : Assume Condition given before holds, then :

$$\mathbb{E}\Big[\int_0^T \int_0^\infty [g(u, \mathcal{Z}(u) + \mathcal{R}) - g(u, \mathcal{Z}(u))]\nu(d\mathcal{R})du\Big] \leq \infty \tag{27}$$

Lemma : If conditions given before are satisfied, then $g(t, \mathcal{Z})$ belongs to the domain of the generator and satisfies the following equation :

$$\begin{cases} \partial_t g(t,\mathcal{Z}) + \gamma \Pi(\mathcal{Z}) g(t,\mathcal{Z}) + \big(\boldsymbol{C}\mathcal{Z} + \mathcal{D}, \partial_{\mathcal{Z}} g(t,\mathcal{Z})\big)_{\mathcal{H}_\alpha} + \frac{1}{2}\mathrm{Tr}\big[\boldsymbol{Q}^c \partial^2_{\mathcal{Z}\mathcal{Z}} g(t,\mathcal{Z})\big] + \\ + \int_{\mathcal{H}_\alpha} \{h(t,\mathcal{Z}+\mathcal{R}) - g(t,\mathcal{Z}) - 1_{\{|\mathcal{R}|_{\mathcal{H}_\alpha}<1\}}((\mathcal{R}, \partial_{\mathcal{Z}} g(t,\mathcal{Z}))_{\mathcal{H}_\alpha} - \}\boldsymbol{\nu}(d\mathcal{R}) = 0, \\ (t,\mathcal{Z}) \in [0,T) \times \mathbb{R} \times \mathcal{H}_\alpha, \; g(T,\mathcal{Z}) = 1, \mathcal{R} \in \mathcal{H}_\alpha. \end{cases} \tag{28}$$

Under assumptions given above the solution of the reduced HJB is given by $h(t, \mathcal{Z}) = g(T - t, \mathcal{Z})$, i.e. :

$$h(t, \mathcal{Z}) = \mathbb{E}\Big[\exp\Big(\int_t^T \gamma \Pi(\mathcal{Z}(u)) du \Big) \Big]. \tag{29}$$

Remark : in order to show that the function proposed is really a solution of the HJB equation we must prove the Frechet differentiability with respect to $\mathcal{Z}$. This is possible, but there are a few technicalities that cannot be fully developed in the few minutes left.

THANKS FOR YOUR KIND ATTENTION ! ! !

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